

# Tropicalizing abelian covers of algebraic curves

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## Abstract

In this paper we will study abelian morphisms  $\mathcal{C} \rightarrow \mathcal{D}$  of semistable curves over a complete discrete valuation ring and the tropical geometry behind them. A semistable Riemann-Hurwitz formula for these abelian coverings will be proved, which will be based on the Laplacian operator for intersection graphs. We will then reconstruct the Berkovich minimal skeleton of  $\mathcal{C}$  from the minimal skeleton of  $\mathcal{D}$ . This will be done explicitly for hyperelliptic curves, abelian extensions of  $\mathbb{P}^1$  and genus 3 curves. An indication of how to use these methods for genus 4, 5 and 6 will also be given.

## 1 Introduction

In this paper we consider the following problem. Suppose we have a finite covering of two smooth, geometrically connected curves over the quotient field  $K$  of a complete discrete valuation ring  $R$ , written as

$$\phi : C \rightarrow D.$$

We suppose in this paper that this covering is *finite abelian* with Galois group  $\mathbb{Z}/q\mathbb{Z}$  for some prime  $q$  that is not equal to the characteristic  $p$  of the residue field of  $K$ . We suppose that we have found a semistable model  $\mathcal{D}$  of  $D$  over  $R$ , with intersection graph  $\mathcal{G}(\mathcal{D})$ . We then want to solve the problem:

**Given the semistable reduction graph  $\mathcal{G}(\mathcal{D})$  of  $D$ , what is the semistable reduction graph  $\mathcal{G}(\mathcal{C})$  of  $C$ ?**

The answer is given in two steps: we first determine the exact influence of the ramification points of  $\phi$  on the reduction graph of  $C$ . The next step is to determine the possible options when the covering is *unramified*. The latter case then gives rise to so-called *étale coverings of graphs*. The type of covering obtained depends heavily on the type of torsion point in the Néron model  $\mathcal{J}$  of the Jacobian that was used to create it. In fact, we will show that the unramified coverings naturally decompose into coverings coming from the component group  $\mathcal{J}/\mathcal{J}^0$  and coverings coming from the identity component  $\mathcal{J}^0$ .

The main driving power behind this paper is a classical theorem by Liu and Lorenzini that tells us how to calculate semistable models for Galois coverings whenever the characteristic of the residue field doesn't interfere. We will give these coverings a special name ("disjointly branched coverings")

and study them in detail.

In studying these morphisms, we find various criteria that tell us about the local ramification on the components. Most of these techniques are based on the Laplacian operator  $\Delta$  on the intersection graph  $\mathcal{G}(\mathcal{D})$  of  $D$ , for some semistable model  $\mathcal{D}$ . This allows us to predict local extensions in terms of graph potentials.

From the beginning, the main goal of the paper was to have some kind of technique to handle genus 3 curves. For genus 1 and genus 2 curves, one naturally has a hyperelliptic involution that allows one to determine the semistable reduction type, at least in principal. We will review this material in terms of our techniques in section 7. For genus 3, one encounters for the first time curves that do not have a hyperelliptic involution. One is then naturally led to consider morphisms of higher degree. These morphisms are however not necessarily Galois, so one instead considers the Galois closure of these morphisms. For genus 3 for instance, one naturally obtains a morphism of degree 3 to  $\mathbb{P}^1$ . This degree 3 map can then have Galois group  $A_3$  or  $S_3$ , which are both solvable. The techniques of this paper are then directly applicable.

In section 8, we will show how to use our techniques for these nonabelian morphisms  $C \rightarrow \mathbb{P}^1$  with a *solvable Galois group*, beginning with a very easy example coming from elliptic curves. This already gives a *new proof of semistability* for elliptic curves, including the usual criterion with  $j$ -invariants. The proof is rather clumsy compared to the usual proof using the hyperelliptic covering, but it already shows some of the flavors one encounters in dealing with general abelian coverings. We then move over to the genus 3 case, where we study a certain canonical degree 3 morphism. We show that the curve corresponding to the Galois closure of this morphism can have at most genus 12. We end the paper with a quick look towards genus 4, 5 and 6, where based on general theorems, we find that our techniques also work.

The paper is divided into the following sections:

- Section 1: Introduction
- Section 2: Notation and terminology
- Section 3: Semistable models and Galois quotients.
- Section 4: Tropical Jacobians, Intersection Theory and Néron models.
- Section 5: Reducing Cartier Divisors, Abelian Covers and Riemann-Hurwitz.
- Section 6: Étale Morphisms of Graphs.
- Section 7: Abelian Extensions of  $\mathbb{P}^1$ .
- Section 8: Graphs for solvable extensions,  $S_3$  and genus 3.

We now give some remarks that should be kept in mind for the rest of the paper.

1. *We will avoid issues of inseparability wherever we can.* We will do this by requiring that the order of the Galois group is not divisible by the characteristic of the residue field  $k$ .
2. *We will only consider coverings of prime degree  $q$ .* One can in fact generalize most of the ideas of this paper to more general abelian covers. We stick to the degree  $q$ -case for simplicity.

3. *We will give a lot of examples to keep everything concrete.* Most of these will be contained in sections 7 and 8. We will also give examples in between the theoretical expositions. Furthermore, several proofs of statements that could have been quoted from a suitable source are included to keep the text more self-contained. The advanced reader is advised to read over these parts.

## 2 Notation and terminology

Let  $R$  be a complete discrete valuation ring with quotient field  $K$  and residue field  $k$ . Throughout the paper  $\pi$  will be a uniformizer. We will assume that the residue field is algebraically closed. A **curve** over  $K$  will be an algebraic variety whose irreducible components are of dimension 1. A **fibered surface** over  $S := \text{Spec } R$  (in short: over  $R$ ) is an integral, projective, flat  $R$ -scheme  $\tau : \mathcal{C} \rightarrow S$  of dimension 2. The generic fiber of  $\mathcal{C}$  will be denoted by  $\mathcal{C}_\eta$  and the special fiber by  $\mathcal{C}_s$ . An **arithmetic surface** is a fibered surface over  $S$  that is regular. A **model** of a curve  $C$  over  $K$  is a normal fibered surface  $\mathcal{C} \rightarrow S$  together with an isomorphism  $f : \mathcal{C}_\eta \simeq C$ . We will assume throughout the paper that *curves  $C$  over  $K$  are smooth and geometrically connected*. A model  $\mathcal{C}$  is said to be **semistable** if the special fiber  $\mathcal{C}_s$  is reduced and has only ordinary double points as its singularities. We will adopt the terminology of [1] and say that the model  $\mathcal{C}$  is **strongly semistable** if in addition to semistability the irreducible components of  $\mathcal{C}_s$  are all smooth.

**Example 2.1.** Let  $A := R[x, y]/I$  where  $I$  will be specified.  $A$  will be considered to be an open affine subscheme of a projective model of some curve. We will assume that  $\text{char}(k) \neq 2$  below.

1. (Example that is not flat) Let  $I = (\pi(y^2 - x^3 - 1))$ . Then the generic fiber is an elliptic curve and the special fiber is  $k[x, y]$ . This is obviously not what we want.
2. (Example that is not strongly semistable) Take  $I = (y^2 - x^3 - x^2 - \pi)$ . The corresponding model is semistable. The special fiber is singular, with one component  $y^2 = x^3 + x^2$ , so this is an example of a semistable model that is not strongly semistable.
3. (Example that is strongly semistable) Take  $I = (y^2 - f)$  where

$$f = x(x - \pi)(x + 1)(x + 1 - \pi)(x + 2)(x + 2 - \pi).$$

The corresponding projective model is then strongly semistable. Its generic fiber is a genus 2 curve. The special fiber has two components intersecting each other in three points.

**Definition 2.1. (Dual Intersection Graph)** Let  $\mathcal{C}$  be a strongly semistable model for a curve  $C$  over  $K$ . Let  $\{\Gamma_1, \dots, \Gamma_r\}$  be the set of irreducible components. We define the dual intersection graph  $\mathcal{G}(\mathcal{C})$  of  $\mathcal{C}$  to be the finite graph whose vertices  $v_i$  correspond to the irreducible components  $\Gamma_i$  of  $\mathcal{C}_s$  and whose edges correspond to intersections between components. The latter means that we have one edge for every point of intersection.

**Example 2.2.** The third example above has as its intersection graph two vertices with three edges between them. One can find the graph in Figure 1.

We will also want to keep track of the genera of the components. We will do this by assigning to every vertex in the dual intersection graph its associated genus. For later purposes, we define

$$w(v_i) := g(\Gamma_i).$$



Figure 1: Covering of graphs in Example 2.1.3.

Whenever we draw the graph of a certain curve, we will write the genera next to the components in question. Whenever the component has genus 0, we will omit the zero. We have the following

**Theorem 2.1.** *Let  $\mathcal{C}$  be a strongly semistable model for a smooth curve  $C$  over  $K$  with intersection graph  $G$ . Let  $\beta(G)$  be the Betti number of  $G$  and let  $p_a(\mathcal{C}_s)$  be the arithmetic genus of  $\mathcal{C}_s$ . We then have*

$$p_a(\mathcal{C}_s) = \beta(G) + \sum_{1 \leq i \leq r} w(v_i).$$

*Proof.* See [[9], page 511]. □

**Example 2.3.** In Example 2.1.3 considered earlier, we have that the Betti number is  $3 - 2 + 1 = 2$ . The curve  $C$  has genus 2 (which can be seen by applying Riemann-Hurwitz to the map corresponding to  $(x, y) \rightarrow x$ ), so the two coincide.

In the upcoming sections we will quite often consider graphs of semistable models that are obtained by subdividing the graph. To do this, we will introduce a notion of *length* on our graphs, making them **weighted graphs**, as in [1].

**Lemma 1.** *Let  $\mathcal{C} \rightarrow S$  be a strongly semistable model with  $S$  as above. Let  $x \in \mathcal{C}_s$  be a singular point. We then have*

$$\hat{\mathcal{O}}_{\mathcal{C},x} \simeq R[[u, v]]/(uv - c)$$

for some  $c \in \mathfrak{m}$ .

*Proof.* See [[9], page 514] for the proof. Note that our ring  $R$  is complete, so we don't need to make an étale extension. □

**Definition 2.2.** We define the **thickness** or length of a singular point on  $\mathcal{C}_s$  to be  $v(c)$ .

We define the length of any edge in our dual intersection graph now to be the length of the corresponding singular point on  $\mathcal{C}_s$ . This makes our graph a metric graph.

We can now also *subdivide* our intersection graph by putting vertices on edges at certain rational distances away from the other vertices. These subdivisions then correspond to semistable models (by a combination of [[9],page 515] and blowing down components). For the Berkovich version, the

reader is directed to [[3]] and specifically Chapters 3 and 4.

Let us recall that for a surjective proper  $\mathcal{X} \rightarrow \operatorname{Spec}(R)$  with  $R$  Henselian, we have a reduction map on the closed points of the generic fiber. Let  $X = \mathcal{X}_\eta$  and let  $X^0$  be the closed points of  $X$ .

**Definition 2.3.** We define the **reduction map**  $r_{\mathcal{X}} : X^0 \rightarrow \mathcal{X}_s$  by

$$r_{\mathcal{X}} : x \mapsto \overline{\{x\}} \cap \mathcal{X}_s.$$

*Remark 2.1.* We note first that  $r_{\mathcal{X}}$  is surjective by [[9], Proposition 1.36., page 468]. Note also that in the definition of the reduction map, one needs the ring  $R$  to be Henselian because otherwise there could be multiple reduction points. One can consider the example

$$\mathcal{X} = \operatorname{Spec}(\mathbb{Z}_{(5)}[x]/(x^2 + 1)) \rightarrow \operatorname{Spec}(\mathbb{Z}_{(5)}[x]/(x^2 + 1)),$$

where  $\mathbb{Z}_{(5)}$  is the localization of  $\mathbb{Z}$  at  $(5)$ . One then takes the closed point of the generic fiber to be the generic point. There are then two possible reductions:  $(x - 1, 5)$  and  $(x - 2, 5)$ . Note that if we instead take the 5-adic ring in the above example, then  $\mathcal{X}$  has two connected components.

**Definition 2.4.** Let  $\mathcal{X} \rightarrow \operatorname{Spec}(R)$  be irreducible, surjective and proper. Let  $\tilde{x}$  be a closed point of  $\mathcal{X}_s$ . Define

$$X_+(\tilde{x}) := r_{\mathcal{X}}^{-1}(\tilde{x}).$$

This is known as the *formal fiber* of  $\tilde{x}$ .

*Remark 2.2.* For semistable models, these formal fibers are naturally isomorphic to open annuli and spheres, where one takes an absolute value corresponding to the valuation on  $R$ . These notions play an important role in analytic theories of semistability, to name a few: Rigid geometry, Formal  $R$ -schemes and Berkovich spaces. In the Berkovich theory one also has formal fibers for points that are not necessarily closed in  $\mathcal{X}_s$ : for instance a generic point of a component. These are known as the type 2 points for curves.

**Example 2.4.** Let  $\mathcal{C} = \operatorname{Proj} R[X, T, W]/(XT - \pi^n W^2)$  with open affine  $U = \operatorname{Spec}(R[x, t]/(xt - \pi^n))$  where  $x = \frac{X}{W}$  and  $t = \frac{T}{W}$ . Let  $C$  be its generic fiber. Let  $\tilde{x} = (x, t, \pi)$ . Note that  $\tilde{x}$  is not a regular point. We then have that

$$C_+(\tilde{x})(K) = \{a \in K : |\pi|^n < |a| < 1\}.$$

That is, it is an open annulus. See [[9], page 471] for the details.

### 3 Semistable Models and Galois quotients

In this section we will review an important theorem regarding coverings of semistable models and we will study some of its properties. We will give these coverings a special name: "**disjointly branched coverings**". This will then lead us to Galois actions on semistable models and the corresponding theory of decomposition groups. This will be very reminiscent of the usual study of Galois extensions of number fields.

#### 3.1 Theorem on semistable covers

Throughout this paper, we'll be making great use of a "classical" theorem on covers of semistable curves. This theorem also gives a practical way of explicitly calculating a lot of semistable reduction graphs. The theorem itself states that the normalization of a semistable model  $\mathcal{D}$  of a curve  $D$  that separates the branch points (in the special fiber!) of a finite cover  $C \rightarrow D$ , will yield a semistable model  $\mathcal{C}$  of  $C$  after some finite extensions. This already gives some intuition why tropical geometry is useful in calculating the intersection graph of a semistable model: points that reduce to the same point on the special fiber will have a relative distance with strictly positive valuation, which is a tropical condition.

**Theorem 1** ([9], Chapter 10, Prop. 4.30). [**Obtaining semistable models from coverings**] Let  $f : C \rightarrow D$  be a finite morphism of smooth, projective geometrically connected curves over  $K$ . Suppose that  $f$  is Galois with group  $G$  of order prime to  $\text{char}(k)$  and that  $D$  admits a semistable model  $\mathcal{D}_0$  over  $R$ . Then the potential stable reduction of  $C$  can be obtained by following the steps below:

1. (**Including branch points**) Let  $B \subset D$  be the branch locus of  $f$ . Take a finite separable extension  $M/K$  to make the points of  $B$  rational over  $M$ . Replace  $\mathcal{D}_0$  by  $\mathcal{D}_0 \times_{\text{Spec}(R)} \text{Spec}(R')$ , where  $R'$  is a discrete valuation ring that dominates  $R$  and has field of fractions  $M$ .
2. (**Separation**) Let  $\mathcal{B}_0$  be the closure of  $B$  in  $\mathcal{D}_0$ . Perform blow-ups at the closed points of  $\mathcal{B}_0$  to obtain a birational morphism  $\phi : \mathcal{D} \rightarrow \mathcal{D}_0$  with  $\mathcal{D}$  semistable such that the closure  $\mathcal{B}$  of  $B$  in  $\mathcal{D}$  is a disjoint union of sections contained in the smooth locus of  $\mathcal{D}$ .
3. (**Normalization**) Let  $\mathcal{C}_0 \rightarrow D$  be the normalization of  $\mathcal{D}$  in  $K(C_M)$ . Let

$$\mathcal{F} = \{\Delta : \Delta \subseteq \mathcal{D}_s \text{ such that either } p_a(\Delta) \geq 1, \text{ or } \Delta \text{ contains at least three points of } \mathcal{B} \cup (\mathcal{D}_s)_{\text{sing}}\}.$$

Let  $e_\Delta$  denote the ramification index  $e_{\Gamma/\Delta}$  for an irreducible component  $\Gamma$  of  $(\mathcal{C}_0)_s$  lying above  $\Delta$ . Set  $e = \text{lcm}\{e_\Delta \mid \Delta \in \mathcal{F}\}$  and  $e = 1$  if  $\mathcal{F} = \emptyset$ .

Then for any extension of discrete valuation rings  $R'' \supseteq R'$  with  $L := \text{Quot}(R'')$  of ramification index divisible by  $e$ , the normalization  $\mathcal{C}$  of  $\mathcal{D}_{R''}$  in  $K(C_L)$  is a semistable model of  $C_L$ .

*Proof.* A proof can be found in [10], Theorem 2.3, for instance. □

Since we want to use this theorem for intersection graphs, we would like to prove that we can find a morphism of *strongly* semistable models.

**Proposition 3.1.** Let  $\phi : \mathcal{C} \rightarrow \mathcal{D}$  be as in Theorem 1. If  $\mathcal{D}$  is *strongly semistable*, then  $\mathcal{C}$  is also strongly semistable.

*Proof.* The proof is mostly based on the following Lemma and its Corollary:

**Lemma 2.** *Let  $\phi : \mathcal{C} \longrightarrow \mathcal{D}$  be a disjointly branched Galois morphism. Then the pre-image of a smooth point consists of smooth points.*

*Proof.* This is the last statement of [[10], Theorem 2.3., page 69]. □

Lemma 2 then implies that

**Corollary 3.1.** For every ordinary double point  $x$  of  $\mathcal{C}$ , the image  $\phi(x)$  is an ordinary double point of  $\mathcal{D}$ .

*Proof.* Indeed, if  $\pi(x)$  is smooth, then there exists a non-smooth point in the pre-image of  $\pi(x)$  (namely  $x$ ). This contradicts Lemma 2. Thus  $\pi(x)$  is non-smooth. Since  $\mathcal{D}$  is semistable, it must be an ordinary double point, as desired. □

Now for the rest of the proof. Suppose that  $x'$  is an ordinary double point in  $\mathcal{C}$ . Let  $\mathfrak{m}'$  be the corresponding maximal ideal on some open affine  $A'$ , which is the integral closure of  $A$  corresponding to an open affine of  $\mathcal{D}$ . Then  $\phi(x')$  is also an ordinary double point by Corollary 3.1. Let  $\mathfrak{m}$  be the corresponding maximal ideal. Since  $\mathcal{D}$  is assumed to be strongly semistable, we can find two distinct prime ideals  $\mathfrak{p}_1$  and  $\mathfrak{p}_2$  (corresponding to two components intersecting each other in  $\phi(x')$ ) in the special fiber such that

$$\mathfrak{p}_i \subset \mathfrak{m}$$

for both  $i$ . By the going-up theorem (which is applicable because  $A \subseteq A'$  is integral), we can find  $\mathfrak{q}_i \subset \mathfrak{m}'$  such that  $\mathfrak{q}_i \cap A = \mathfrak{p}_i$  (note that they are in the special fiber by this condition). But then  $\mathfrak{m}'$  is an intersection point of  $\mathfrak{q}_1$  and  $\mathfrak{q}_2$ . This proves that  $\mathcal{C}$  is strongly semistable, as desired. □

Since we will be mostly in the scenario of Theorem 1 in this paper, we will give these morphisms a special name.

**Definition 3.1.** Let  $\mathcal{C} \longrightarrow \mathcal{D}$  be a Galois morphism of semistable models over  $R$  as obtained from Theorem 1. Suppose in addition that  $\mathcal{D}$  is strongly semistable (which by Proposition 3.1 ensures that  $\mathcal{C}$  is also strongly semistable). We then call these morphisms: disjointly branched Galois morphisms of strongly semistable models, or in short: **disjointly branched morphisms**.

Let us now make some remarks about Theorem 1 that should be kept in mind throughout the paper.

*Remark 3.1 (About the branch points).* Since we assumed the residue field  $k$  to be algebraically closed, we only have to take *ramified extensions* of  $K$  here.

For the extensions in the third step, we find that the extensions are in fact *tame*. Indeed, the ramification index of any component has to divide the order of the Galois group, which is coprime to the characteristic of  $k$ . This then also means that the extensions in the third step are obtained by  $K \subseteq K(\pi^{1/n})$  for some  $n$  with  $(n, \text{char}(k)) = 1$ .

*Remark 3.2 (About the closure of  $B$ ).* The closure of  $B$  in  $\mathcal{D}$  can be computed using the *reduction* map. For any closed point  $x$  of the generic fiber, we have that

$$\overline{\{x\}} = \{x, r_{\mathcal{D}}(x)\}.$$

That is, we take the point  $x$  together with its reduction. The condition here is that the reductions of the branch points are *disjoint* and that they reduce to nonsingular points.

$$\begin{array}{ccc}
K(C) & \longrightarrow & K(\overline{C}) \\
\uparrow & \nearrow & \\
K(\mathbb{P}^1) & & 
\end{array}$$

Figure 2: Commutative diagram corresponding to a non-Galois morphism  $K(\mathbb{P}^1) \rightarrow K(C)$ .

*Remark 3.3 (Caveat about the characteristic).* The condition on the characteristic of the residue field is to avoid issues of separability in the special fiber. Indeed, one encounters residue fields that are *not perfect* and as such there exists extensions of Frobenius type. There is a way to address these cases as well, see [19].

*Remark 3.4 (Galois action).* In the theorem we take the normalization of a certain normal integral scheme in a Galois extension. The resulting scheme  $\mathcal{C}$  then actually has a natural Galois action such that  $\mathcal{C}/G = \mathcal{D}$ . These Galois actions will be reviewed shortly.

The theorem will be used in the following way already indicated in [9]: we will assume that we have a covering  $f : C \rightarrow \mathbb{P}^1$  of degree  $n$ . This gives us a separable extension of function fields  $K(\mathbb{P}^1) \rightarrow K(C)$ . This morphism is not in general *Galois*, so we have to take the Galois closure. This gives us a possibly very large extension of degree  $\leq n!$  with the commutative diagram as in Figure 3.1. The idea is then to first find a semistable model of  $\overline{C}$ . Projecting down the semistable model of  $\overline{C}$  onto  $C$  will then give the semistable model of  $C$  by [[9], Chapter 10, Prop. 3.48].

The author of [9] pointed out its use for calculating cyclic covers of  $\mathbb{P}^1$ , but it can actually be used to calculate general covers. We will give plenty of examples in the subsequent chapters.

*Remark 3.5 (A few tricks for calculating reduction graphs).* A brute-force computation of the normalization might not always be insightful, so we will quite often do the following in this paper. To obtain the reduction type we only calculate the normalization at codimension 1 primes. This usually suffices to calculate the pre-image of the intersection points: one subdivides the underlying graph into more components (which will quite often involve a ramified base extension), which then will yield the number of intersection points lying in the pre-image. There might be some non-trivial mixing however, which occurs in the *totally unramified* case. See Chapter 6 for this case.

## 3.2 Galois Quotients

In the last section we defined disjointly branched morphisms. The corresponding semistable models (derived from Theorem [9]) have a natural Galois action on them. In this section, we will review some facts about quotient schemes for finite Galois groups that act on a scheme  $X$ . We will quickly specialize to the strongly semistable case, where we consider the problem of Galois actions on graphs. We will also introduce decomposition groups for vertices (and edges) in the intersection graph and review some basic material regarding these.

We will follow [5] and [[9], Exercises 2.14, 2.3.21 and 3.3.23]. Let  $X$  be a scheme with a finite group  $G$  acting on  $X$ . This means that we have a group homomorphism

$$G \rightarrow \text{Aut}(X).$$



The *quotient scheme* is then defined by a universal property that we will repeat here. The quotient scheme of  $X$  under  $G$  is a scheme  $Y$  with the following properties:

1. There is a morphism  $p : X \rightarrow Y$ .
2. We have  $p = p \circ \sigma$  for every  $\sigma \in G$ .
3. Any morphism of schemes  $f : X \rightarrow Z$  satisfying  $f = f \circ \sigma$  for every  $\sigma$  factors in a unique way through  $p$ . This means that there exists a unique morphism  $\tilde{f} : Y \rightarrow Z$  such that  $f = \tilde{f} \circ p$ .

Let us consider the affine case first. The following proposition is from [5].

**Proposition 3.2. [Affine quotients]** Let  $A$  be a ring with a finite group action  $G \rightarrow \text{Aut}(A)$ . Let  $B = A^G$  be the invariants,  $X = \text{Spec}(A)$ ,  $Y = \text{Spec}(B)$  and  $p : X \rightarrow Y$  the canonical morphism. Then

1.  $A$  is integral over  $B$  (and the morphism  $p$  is thus integral).
2. The morphism  $p$  is surjective, its fibres are the orbits under  $G$ , the topology of  $Y$  is the quotient of the topology on  $X$ .
3. Let  $x \in X$ ,  $y = p(x)$ ,  $G_x$  the stabilizer of  $x$ . Then  $k(x)$  (the residue field of  $x$ ) is a normal algebraic extension of  $k(y)$  and the homomorphism  $G_x \rightarrow \text{Gal}(k(x)/k(y))$  is surjective.
4.  $(Y, p)$  is the quotient scheme of  $X$  by  $G$ .
5. The natural morphism

$$\mathcal{O}_Y \rightarrow p_*(\mathcal{O}_X)^G$$

of sheaves is an isomorphism.

*Proof.* This is almost a word-by-word translation of [[5], Exposé V, Proposition 1.1. and Corollary 1.2.].  $\square$

Let us now generalize a little bit. We will consider what Grothendieck calls "admissible actions".

**Definition 3.2.** Let  $G$  be a finite group acting on a scheme  $X$ ,  $p : X \rightarrow Y$  an *affine* invariant morphism such that

$$\mathcal{O}_Y \rightarrow p_*(\mathcal{O}_X)^G.$$

This action is then called an **admissible action**.

**Proposition 3.3.** Let  $G$  give an admissible action on  $X$ . Then the conclusions from Proposition 3.2 are still valid. In particular, we have  $Y = X/G$ .

*Proof.* This is [[5], Exposé V, Proposition 1.3.].  $\square$

**Corollary 3.2.** Let  $G$  give an admissible action on  $X$ . Then for any open set  $U \subset Y$ , we have that  $U$  is the quotient of  $p^{-1}(U)$  under  $G$ .

*Proof.* See [[5], Exposé V, Corollary 1.4.].  $\square$

**Proposition 3.4.** Let  $G$  be a finite group acting on a scheme  $X$ . Then  $G$  gives an admissible action if and only if there is an open affine cover  $\{U_i\}$  of  $X$  such that each  $U_i$  is invariant under  $G$ .

*Proof.* See [[5], Exposé V, Proposition 1.8.]. □

Let us now focus on *normal integral* schemes.

**Proposition 3.5.** Let  $Y$  and  $X$  be normal integral schemes. Suppose that we have a finite surjective integral morphism  $f : X \rightarrow Y$  such that  $K(X)/K(Y)$  is a finite Galois extension with Galois group  $G$ . Then  $X$  is the normalization of  $Y$  in  $K(X)$ . We have an action of  $G$  on  $X$  with  $X/G = Y$ . Furthermore, this action on  $X$  is transitive.

*Proof.* The fact that  $X$  is the normalization follows from [[9], page 120, Proposition 1.22]. The normalization naturally comes with a group action, stemming from the fact that on affines we have that if  $a' \in A'$  is integral over  $A$ , then  $\sigma(a')$  is also integral over  $A$  for any  $\sigma \in G$ . Now consider the chain

$$A' \supseteq (A')^G \supseteq A.$$

We have that  $(A')^G = A$ . Indeed, if  $a' \in (A')^G$ , then  $a' \in K(Y)$  and  $a'$  is integral over  $A$ . Since  $A$  is integrally closed in  $K(Y)$ , we have that  $a' \in A$ , as desired. This then yields  $\mathcal{O}_Y = p_*(\mathcal{O}_X)$  and thus  $Y = X/G$  by Proposition 3.3. For transitivity, see [[9], page 546, Lemma 4.34]. □

### 3.2.1 Quotients and special fibres

From Proposition 3.5 we see that in the case of a disjointly branched Galois morphism  $\mathcal{C} \rightarrow \mathcal{D}$ , we also have that  $\mathcal{C}/G = \mathcal{D}$ . Let us try to specialize this equality to the special fiber. Let  $\mathcal{I}$  be the ideal sheaf of the special fiber  $\mathcal{C}_s$ . That is,  $\mathcal{I} = \pi\mathcal{O}_{\mathcal{C}}$ . We then have an exact sequence

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_{\mathcal{C}} \rightarrow \mathcal{O}_{\mathcal{C}_s} \rightarrow 0$$

of sheaves of abelian groups, where  $\mathcal{O}_{\mathcal{C}}$  and  $\mathcal{O}_{\mathcal{C}_s}$  are the structure sheaves. These sheaves have a natural  $G$ -action on them. We consider the cohomology group  $H^1(G, \mathcal{I})$  coming from group cohomology. Since  $|G|$  is invertible in  $\mathcal{O}_{\mathcal{C}}$  (this is the condition on the characteristic being prime to the Galois group) we have that

$$H^1(G, \mathcal{I}) = (1).$$

Indeed,  $|G|$  is invertible in  $\mathcal{O}_{\mathcal{C}}$  and thus also in  $H^1(G, \mathcal{I})$ . We then have that  $|G|$  annihilates  $H^1(G, \mathcal{I})$  by [[16], Chapter VIII, Section 2, Corollary 1] and thus  $H^1(G, \mathcal{I}) = (1)$ .

We now use the long exact sequence of group cohomology, which yields

$$(\mathcal{O}_{\mathcal{C}_s})^G = (\mathcal{O}_{\mathcal{C}}/\mathcal{I})^G = \mathcal{O}_{\mathcal{D}}/(\mathcal{I})^G = \mathcal{O}_{\mathcal{D}_s},$$

implying that

$$\mathcal{C}_s/G = \mathcal{D}_s.$$

<sup>(1)</sup> We thus also have a quotient on the special fiber.

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<sup>1</sup>This argument can also be found in [[10], Remark 1.7].

### 3.3 Galois extensions of Dedekind domains

We will now describe the disjointly branched morphisms  $f : \mathcal{C} \rightarrow \mathcal{D}$  locally on codimension 1 primes. We will try to maneuver ourselves into a position where we can use the usual theorems on Dedekind domains. Consider the ring extension on an open affine  $U = \text{Spec}(A)$  of  $\mathcal{D}$  given by

$$A \subseteq A'$$

corresponding to  $f$ . Here  $A'$  is the integral closure of  $A$  in  $K(C)$ . Let  $\mathfrak{p}$  be a prime of  $A$ . Recall that  $f$  is finite, meaning that the ring extension is also finite. We can take the base extension

$$A_{\mathfrak{p}} \rightarrow A_{\mathfrak{p}} \otimes A' = (A')_{\mathfrak{p}}$$

of the map  $A \rightarrow A_{\mathfrak{p}}$ . Since finiteness is preserved under base extensions, we have that this ring extension is again finite. Taking another base extension, this time corresponding to  $A_{\mathfrak{p}} \rightarrow A_{\mathfrak{p}}/\mathfrak{p}$ , we obtain that the ring extension

$$A_{\mathfrak{p}}/\mathfrak{p} \rightarrow (A')_{\mathfrak{p}}/\mathfrak{p}$$

is again finite. This just means that  $(A')_{\mathfrak{p}}/\mathfrak{p}$  is a finite vector space over the field  $A_{\mathfrak{p}}/\mathfrak{p}$ . We thus have that it's an Artinian ring, meaning that there are only finitely many prime ideals.<sup>2</sup> These prime ideals correspond exactly to the prime ideals of  $A'$  above the prime ideal  $\mathfrak{p}$ . A small reminder: localization commutes with normalization, so we can write

$$(A')_{\mathfrak{p}} = (A_{\mathfrak{p}})'.$$

Let us now restrict ourselves to the  $\mathfrak{p}$  such that  $\dim A_{\mathfrak{p}} = 1$ . In this case, we have that  $A_{\mathfrak{p}}$  is a normal Noetherian integral domain of dimension 1. Or in other words, we have that  $A_{\mathfrak{p}}$  is a *Dedekind domain*. Since  $A'_{\mathfrak{p}}$  is finite over  $A_{\mathfrak{p}}$ , we have that  $1 = \dim(A_{\mathfrak{p}}) = \dim(A'_{\mathfrak{p}})$ . Since  $A'$  is normal, any localization of  $A'$  is also normal. Thus  $A'_{\mathfrak{p}}$  is also a Dedekind domain. This puts us back into the usual framework of algebraic number theory: finite ring extensions of Dedekind domains. One major difference of course with the number field theory is that the residue fields involved can be nonperfect.

We recall some definitions and theorems from algebraic number theory. For the rest of the section,  $A$  and  $A'$  are Dedekind. We say that  $\mathfrak{q} \in \text{Spec}(A')$  *divides*  $\mathfrak{p} \in \text{Spec}(A)$  if  $\mathfrak{q} \cap A = \mathfrak{p}$ . We define the ramification index of  $\mathfrak{q}$  over  $\mathfrak{p}$  by

$$e_{\mathfrak{q}/\mathfrak{p}} = \dim_{A/\mathfrak{p}}(A'_{\mathfrak{q}}/\mathfrak{p}A'_{\mathfrak{q}}).$$

We will also write this as  $e_{\mathfrak{q}}$ . We then have a decomposition of prime ideals

$$\mathfrak{p} = \prod_{i=1}^r \mathfrak{q}_i^{e_{\mathfrak{q}_i}}.$$

For each  $\mathfrak{q}$  dividing  $\mathfrak{p}$  we have a finite extension of residue fields

$$A_{\mathfrak{p}}/\mathfrak{p} \rightarrow A'_{\mathfrak{q}}/\mathfrak{q}$$

with finite degree

$$f_{\mathfrak{q}/\mathfrak{p}} = [A'_{\mathfrak{q}}/\mathfrak{q} : A_{\mathfrak{p}}/\mathfrak{p}].$$

We will also let  $g_{\mathfrak{p}}$  be the number of primes in  $\text{Spec}(A')$  dividing a given  $\mathfrak{p}$ . For a *Galois* extension, we then have the following

---

<sup>2</sup>We are just expressing the fact that a finite morphism of schemes is quasi-finite of course.

**Proposition 3.6.** Suppose that the extension  $A \subseteq A'$  with fraction fields  $K \subset L$  is Galois with Galois group  $G$ . Let  $n$  be its order. Then the integers  $e_{\mathfrak{q}}$  and  $f_{\mathfrak{q}}$  depend only on  $\mathfrak{p}$ . If we denote them by  $e_{\mathfrak{p}}$  and  $f_{\mathfrak{p}}$ , then

$$n = e_{\mathfrak{p}} f_{\mathfrak{p}} g_{\mathfrak{p}}. \quad (1)$$

*Proof.* See [[16], page 20].  $\square$

We now define the so-called **decomposition groups** and **inertia groups**. Let  $\mathfrak{q}$  be a prime of  $\text{Spec}(A')$ . Consider

$$D_{\mathfrak{q}} = \{\sigma \in G : \sigma(\mathfrak{q}) = \mathfrak{q}\}.$$

We sometimes denote it by  $D$ . This is the **decomposition group** of  $\mathfrak{q}$ . By the orbit-stabilizer theorem, we have that the index of  $D$  in  $G$  is equal to the number  $g_{\mathfrak{p}}$ . Let us write

$$\begin{aligned} \bar{L} &= A'_{\mathfrak{q}}/\mathfrak{q}, \\ \bar{K} &= A_{\mathfrak{p}}/\mathfrak{p}. \end{aligned}$$

For any  $\sigma \in D$ , we have a natural  $\bar{K}$ -automorphism  $\bar{\sigma}$  of  $\bar{L}$  by passing to the quotient. This gives a homomorphism

$$\rho : D \longrightarrow G(\bar{L}/\bar{K})$$

whose kernel is by definition the **inertia group** of  $\mathfrak{q}$ , denoted by  $I_{\mathfrak{q}}$  or just  $I$ .

**Proposition 3.7.** The following properties are true.

1. The residue extension  $\bar{L}/\bar{K}$  is normal and the homomorphism

$$\rho : D \longrightarrow G(\bar{L}/\bar{K})$$

defines an isomorphism  $D/I \simeq G(\bar{L}/\bar{K})$ .

2. If  $\bar{L}/\bar{K}$  is separable, then it is a Galois extension with Galois group  $D/I$ . We then have  $[L : L^I] = e$ ,  $[L^I : L^D] = f$  and  $[L^D : K] = g$ . Here  $L^H$  for any subgroup  $H$  of  $G$  means the invariant field under  $H$ .

*Proof.* See [[16], page 23].  $\square$

*Remark 3.6.* The residue extension  $\bar{L}/\bar{K}$  is separable in the following cases:

1.  $\bar{K}$  is perfect (which will generally not be the case for us, unless the residue field  $k$  has characteristic zero).
2. The order of the inertia group  $I$  is prime to the characteristic  $p$  of the residue field  $\bar{K}$ .

In the case of a disjointly branched Galois morphism we naturally have case (2). This implies that  $\bar{L}/\bar{K}$  is separable and thus we can use Proposition 3.7.

*Remark 3.7.* For finite morphisms of semistable models, we naturally have that  $e_{\mathfrak{q}/\mathfrak{p}} = 1$ , for any prime  $\mathfrak{p}$  of codimension 1, with  $\mathfrak{q}$  a divisor. This is because  $t$  is a uniformizer in both. We therefore have

$$|G| = f_{\mathfrak{q}/\mathfrak{p}} \cdot g_{\mathfrak{q}/\mathfrak{p}}.$$

*Remark 3.8.* We would like to point out to the reader that we now have two notations that are very similar. On the one hand for an edge  $e$  with corresponding prime  $\mathfrak{p}$  of  $\mathcal{D}$ , we have the "splitting" indices

$$g_{\mathfrak{q}/\mathfrak{p}},$$

which just tell us the orders of the decomposition groups. On the other hand, in a very natural way we have that our primes are curves. We thus also have the genus

$$g(\mathfrak{p})$$

to our disposal. We will sometimes write  $a(\mathfrak{p})$  (as in: abelian rank) or  $g(\Gamma)$  for the arithmetic genus. Whenever we mean the order of the decomposition group, we will write it as a subscript.

### 3.3.1 Decomposition groups

In this section we will quickly look at decomposition groups for more general points. In our set-up we will consider the action of a finite group  $G$  on a quasi-projective scheme  $X$  over a locally Noetherian scheme  $S$ . Note that for these schemes the quotient  $X/G$  automatically exists in the category of schemes. This greater generality allows us to define *decomposition groups for edges* in the intersection graph of a strongly semistable model in the next section. We will follow [9] and [5].

**Definition 3.3 (Decomposition groups).** Let  $X$  be a scheme with an admissible group action of a finite group  $G$  on it. Let  $x$  be any point of  $X$ . We define

$$D_x := \{g \in G : gx = x\}$$

to be the *decomposition group* of  $x$ .

Consider the quotient morphism  $p : X \rightarrow X/G =: Y$ . Let  $y := p(x)$ . We have a natural morphism

$$\mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$$

and in fact we have  $\mathcal{O}_{Y,y} = (\mathcal{O}_{X,x})^G$ , which follows from  $\mathcal{O}_Y = p_*(\mathcal{O}_X)^G$  (see [[5], page 92]). The decomposition group  $D_x$  naturally acts on the residue field of  $k(x)$ . In fact, we have a homomorphism

$$D_x \rightarrow \text{Aut}(k(x)/k(y))$$

just as before.

**Definition 3.4 (Inertia groups).** Consider the homomorphism  $D_x \rightarrow \text{Aut}(k(x)/k(y))$ . We define the *inertia group*  $I_x$  of  $x$  to be the kernel of this homomorphism. As a subgroup of  $D_x$ , it naturally acts on  $\mathcal{O}_{X,x}$ .

**Lemma 3.** *The morphism  $p : X \rightarrow Y$  is étale at  $x$  if and only if  $I$  acts trivially on  $\mathcal{O}_{X,x}$ . More generally, we have that  $X/I \rightarrow Y$  is étale at the image of  $x$  in  $X/I$ .*

*Proof.* See [[9], page 147]. □

In the next section, we will consider these decomposition groups and inertia groups for closed points on semistable models. In contrast to the previous section, where there were *no* inertia groups for codimension 1 primes, we will find that closed points can have nontrivial inertia subgroups.

### 3.4 Galois action on intersection graphs

In the last section we studied decomposition groups for codimension 1 primes and later on for more general points. Here we will continue to study them for disjointly branched morphisms. We will find that there exists a natural Galois action on the intersection graph. In fact, this action will induce a quotient of intersection graphs  $\mathcal{G}(\mathcal{C})/G = \mathcal{G}(\mathcal{D})$ . We will then see that we can specialize decomposition groups for specializations  $\mathfrak{p}_1 \supseteq \mathfrak{p}_2$  in the form of an injection  $D_{\mathfrak{p}_1} \longrightarrow D_{\mathfrak{p}_2}$ .

So let us consider a disjointly branched morphism  $\phi : \mathcal{C} \longrightarrow \mathcal{D}$  with Galois group  $G$ .

**Lemma 4.** *For any  $\sigma \in G$  and  $x \in \mathcal{C}_s$  an intersection point of  $\Gamma_1$  and  $\Gamma_2$ , we have that  $\sigma(x)$  is an intersection point of  $\sigma(\Gamma_1)$  and  $\sigma(\Gamma_2)$ .*

*Proof.* Let  $\mathfrak{m}$  be the maximal ideal corresponding to  $x$  on some open affine  $U$ . Correspondingly, let  $\mathfrak{p}_i$  be the primes corresponding to  $\Gamma_i$ . The fact that  $x$  is an intersection point of  $\Gamma_1$  and  $\Gamma_2$  can be paraphrased by

$$\mathfrak{m} \supseteq \mathfrak{p}_i$$

for both  $i$ . Letting  $\sigma$  act on the above, we obtain

$$\sigma(\mathfrak{m}) \supseteq \sigma(\mathfrak{p}_i)$$

for both  $i$ , meaning that  $\sigma(x)$  is an intersection point of  $\sigma(\Gamma_1)$  and  $\sigma(\Gamma_2)$ .  $\square$

We thus see that  $\sigma$  acts as an automorphism of graphs: if  $v$  and  $v'$  are joined by an edge  $e$ , then  $\sigma(v)$  and  $\sigma(v')$  are joined by the edge  $\sigma(e)$  by the above lemma. We therefore see that we have a homomorphism

$$G \longrightarrow \text{Aut}(\mathcal{G}(\mathcal{C})).$$

We can now define *decomposition groups* and *inertia groups* for elements of our graph.

**Definition 3.5 (Decomposition and inertia groups for graphs).** Let  $v$  and  $e$  be a vertex and edge respectively of the intersection graph  $\mathcal{G}(\mathcal{C})$ . Let the corresponding points in  $\mathcal{C}$  be given by  $x_v$  and  $x_e$ . We define the *decomposition group* of  $v$  and  $e$  to be  $D_{x_v}$  and  $D_{x_e}$  respectively. Similarly, we define the *inertia groups* of  $v$  and  $e$  to be  $I_{x_v}$  and  $I_{x_e}$ .

Recall now that we have a homomorphism  $G \longrightarrow \text{Aut}(\mathcal{G}(\mathcal{C}))$ . We can therefore consider the quotient of graphs

$$\mathcal{G}(\mathcal{C}) \longrightarrow \mathcal{G}(\mathcal{C})/G.$$

*Remark 3.9.* Unfortunately, we can have that  $\mathcal{G}(\mathcal{C})/G \neq \mathcal{G}(\mathcal{D})$  for semistable models  $\mathcal{C}$  and  $\mathcal{D}$ . Indeed, consider the semistable model given by the equation

$$y^2 = x(x - \pi)(x + 1)(x + 1 - \pi)(x + 2)(x + 2 - \pi),$$

which gives an intersection graph with two vertices and three edges between them. We have a natural Galois action on this model, given by

$$y \longmapsto -y$$

on the coordinate rings. Note that the edges of the intersection graph are invariant under the action, giving

$$I_{e_i} = \mathbb{Z}/2\mathbb{Z}.$$

For the components however, we have that

$$D_{v_i} = (1).$$

We thus see that the quotient graph  $\mathcal{G}(\mathcal{C})/G$  consists of one vertex with three loops. The intersection graph of the quotient  $\mathcal{C}/G$  consists of just one vertex however.

Note that we always have a morphism of graphs

$$\mathcal{G}(\mathcal{C})/G \longrightarrow \mathcal{G}(\mathcal{D})$$

by the universal property of the quotient and the fact that every element of  $\mathcal{G}(\mathcal{D})$  is invariant under  $G$ . From the above example we see that this morphism can be noninjective.

We would now like to prove the following theorem:

**Theorem 3.1.** *Let  $\phi : \mathcal{C} \longrightarrow \mathcal{D}$  be a disjointly branched Galois morphism with Galois group  $G$ . Then*

$$\mathcal{G}(\mathcal{C})/G = \mathcal{G}(\mathcal{D}).$$

*Proof.* The proof relies mostly on Lemma 2, that we will restate here.

**Lemma 5.** *Let  $\phi : \mathcal{C} \longrightarrow \mathcal{D}$  be a disjointly branched Galois morphism. Then the pre-image of a smooth point consists of smooth points.*

Let us also restate Corollary 3.1:

**Corollary 3.3.** *For every ordinary double point  $x$  of  $\mathcal{C}$ , we have that the image  $\phi(x)$  is an ordinary double point of  $\mathcal{D}$ .*

To finish the proof of Theorem 3.1, first note that we already have

$$\mathcal{C}/G = \mathcal{D},$$

a quotient of schemes. We thus only have to show that vertices are mapped to vertices and edges to edges. The first follows from the fact that  $\phi : \mathcal{C} \longrightarrow \mathcal{D}$  maps a codimension 1 prime in the special fibre  $\mathcal{C}_s$  to another codimension 1 prime in  $\mathcal{D}_s$ . The second follows from Corollary 3.3. This gives the theorem.  $\square$

Let us now consider the following problem. Suppose that we again have a disjointly branched morphism

$$\phi : \mathcal{C} \longrightarrow \mathcal{D}$$

with Galois group  $G$ . Let  $H$  be any subgroup of  $G$ . We can consider the subquotient

$$\mathcal{C} \longrightarrow \mathcal{C}/H \longrightarrow \mathcal{D},$$

where we define  $\psi : \mathcal{C} \longrightarrow \mathcal{C}/H$  to be the quotient morphism. Note that this quotient is again semistable, by [[9], Proposition 3.48., page 526].

**Lemma 6.** *The image  $\psi(x)$  of an ordinary double point  $x \in \mathcal{C}_s$  is an ordinary double point of  $\mathcal{C}/H$ .*

*Proof.* Suppose that  $\psi(x)$  is smooth. Then the image of  $\psi(x)$  in  $\mathcal{D}$  is also smooth, by [[10], Proposition 1.6., page 16]. This contradicts Corollary 3.3, concluding the proof.  $\square$

**Lemma 7.** *Let  $\mathcal{C}$ ,  $\mathcal{D}$  and  $H$  be as above. Then*

$$\mathcal{G}(\mathcal{C})/H = \mathcal{G}(\mathcal{C}/H).$$

*Proof.* As in Theorem 3.1, we have a Galois quotient

$$\mathcal{C} \longrightarrow \mathcal{C}/H$$

with Galois group  $H$ . As in that theorem, we see that vertices are mapped to vertices and edges to edges, so the lemma follows.  $\square$

The above lemma allows us to find the intersection graph of quotients by just taking the quotient of the intersection graphs. We will use this quite often in the examples to come.

**Definition 3.6.** Let  $\phi : \mathcal{C} \longrightarrow \mathcal{D}$  be a disjointly branched morphism. We define  $\phi_{\mathcal{G}} : \mathcal{G}(\mathcal{C}) \longrightarrow \mathcal{G}(\mathcal{D})$  to be the induced morphism on intersection graphs. Note that this is well-defined by Theorem 3.1.

*Remark 3.10.* Let us quickly say something about the weights that our vertices in  $\mathcal{G}(\mathcal{D})$  might have had. Recall that these weights correspond to the genera of the corresponding components. For any morphism

$$\Gamma_C \longrightarrow \Gamma_D$$

of components in the special fibre, we assign the genus of  $\Gamma_C$  to the vertex corresponding to it. Throughout the paper, we will assume implicitly that we have done this for every vertex of  $\mathcal{G}(\mathcal{C})$ .

### 3.4.1 Specialization of decomposition groups

We will now consider the following problem. Let us consider the chain

$$\mathfrak{p} \subset \mathfrak{m},$$

where  $\mathfrak{p}$  corresponds to a component in the special fiber and  $\mathfrak{m}$  to an intersection point. We can, in general, not find an injective morphism

$$D_{\mathfrak{m}} \longrightarrow D_{\mathfrak{p}}$$

for semistable models. Indeed, we saw this in Remark 3.9. We will now show that we do obtain such an injection for strongly semistable models.

**Proposition 3.8.** Let  $\phi$  be a disjointly branched morphism with  $\mathfrak{m} \supset \mathfrak{p}$  as above. There is then a canonical injective morphism  $D_{\mathfrak{m}} \longrightarrow D_{\mathfrak{p}}$ .

*Proof.* We will show the following: if  $\sigma$  fixes  $\mathfrak{m}$ , then it also fixes  $\mathfrak{p}$ . Suppose for a contradiction that it doesn't fix  $\mathfrak{p}$ . Then  $\sigma(\mathfrak{p})$  corresponds to a different component. We have  $\sigma(\mathfrak{m}) = \mathfrak{m}$ , so we find

$$\mathfrak{m} \supset \sigma(\mathfrak{p}).$$

This just means (together with  $\mathfrak{m} \supset \mathfrak{p}$ ) that  $\mathfrak{m}$  is an intersection point of the components  $\Gamma$  and  $\sigma(\Gamma)$ , which correspond to  $\mathfrak{p}$  and  $\sigma(\mathfrak{p})$ . We will now find a contradiction using



**Lemma 8.** *Let  $G$  be a finite group acting on a semistable model  $\mathcal{C}$ . Let  $x$  be an ordinary double point of  $\mathcal{C}$ , connecting two components  $\Gamma$  and  $\Gamma'$ . Let  $I$  be the inertia subgroup of  $x$  and let*

$$\pi : \mathcal{C} \longrightarrow \mathcal{C}/I$$

*be the corresponding quotient map. Then  $\pi(x)$  is smooth in  $\mathcal{C}/I$  if and only if there exists an element  $\sigma \in I$  such that*

$$\sigma(\Gamma) = \Gamma'.$$

*Proof.* Let

$$I_0 = \{\sigma \in I : \sigma(\Gamma) = \Gamma\}.$$

By tracing through the proof of [[9], page 527, Proposition 3.48], one finds that the case with  $I_0 \subsetneq I$  corresponds to  $\pi(x)$  being smooth and the case  $I_0 = I$  to  $\pi(x)$  being an ordinary double point. The Lemma then quickly follows.  $\square$

The inclusion  $\mathfrak{m} \supset \sigma(\mathfrak{p})$  will now give us the desired contradiction. Indeed, we see that  $\sigma(\Gamma) = \Gamma'$  and  $\sigma$  is an element of the inertia subgroup of  $x$  (here we use that our residue field  $k$  is algebraically closed). But then  $\pi(x)$  is smooth by Lemma 8. This contradicts Corollary 3.3, as desired.  $\square$

For smooth points  $\mathfrak{m} \supset \mathfrak{p}$ , we also have an injection

$$D_{\mathfrak{m}} \longrightarrow D_{\mathfrak{p}}.$$

This is much easier to prove however, since there is only one component that contains  $\mathfrak{m}$ .

## 4 Tropical Jacobians and Néron models

We will study Tropical Jacobians and their relation to Néron models here. This section can, for the most part, also be found in [1]. We will put more emphasis on the vertical divisors however. These vertical divisors will play an important role in determining the local ramification at the intersection points.

The problem we wish to address here is as follows: we wish to *transport* divisors from our curve  $C$  to a strongly semistable regular model  $\mathcal{C}$  and then to its intersection graph  $G$ . This will require some notions from graph theory and intersection theory. In each of the three settings we have a notion of a *principal* divisor. This will then give us the notion of a "Jacobian" in each scenario.

We will start with intersection graphs and Jacobians on these intersection graphs. Here we will introduce the Laplacian operator. We will then move to intersection theory on  $\mathcal{C}$ , where we will show how to move from divisors on  $\mathcal{C}$  to divisors on the intersection graph. Lastly, we will study how the Néron model of the Jacobian of  $C$  fits into all of this and how we can make sense of the identity component of that Néron model in terms of graph cohomology.

### 4.1 Divisors on graphs and Laplacians

So let  $G$  be a graph, which we will assume to be finite, connected and without loop edges. Let  $V(G)$  be its vertices and  $E(G)$  its edges. We define  $\text{Div}(G)$  to be the free abelian group on the vertices  $V(G)$  of  $G$ . Its elements will be referred to as divisors on  $G$ . Writing  $D \in \text{Div}(G)$  as  $D = \sum_{v \in V(G)} c_v(v)$ , we define the degree map as  $\deg(D) = \sum_{v \in V(G)} c_v$ . We let  $\text{Div}^0(G)$  be the group of divisors of degree zero on  $G$ .

Now let  $\mathcal{M}(G)$  be the group of  $\mathbb{Z}$ -valued functions on  $V(G)$ . Define the **Laplacian operator**  $\Delta : \mathcal{M}(G) \rightarrow \text{Div}^0(G)$  by

$$\Delta(\phi) = \sum_{v \in V(G)} \sum_{e=vw \in E(G)} (\phi(v) - \phi(w))(v).$$

We then define the group of principal divisors to be the image of the Laplacian operator:

$$\text{Prin}(G) := \Delta(\mathcal{M}(G)).$$

*Remark 4.1.* We took the discrete definition of Laplacians from [[1]]. Quite often we will also tacitly use the  $\mathbb{Q}$ -metric graph version by taking ramified extensions and considering divisors on subdivisions of our original graph.

**Definition 4.1 (Tropical Jacobians).** We define the *Tropical Jacobian* of  $G$  to be the group

$$\text{Div}^0(G)/\text{Prin}(G). \tag{2}$$

**Example 4.1.** Suppose we take  $\text{Proj } R[X, Y, W]/(XY - \pi W^2)$  with its usual grading. As before, we have two components intersecting each other in one point. It is now quite easy to see that every divisor of degree zero is in fact principal. Take any  $D$  of degree zero. Then  $D(\Gamma_1) = -D(\Gamma_2)$ . Let us define

$$\begin{aligned} \phi(\Gamma_1) &= 0, \\ \phi(\Gamma_2) &= D(\Gamma_2). \end{aligned}$$

Then  $\phi$  has the right divisor and as such every divisor is principal.

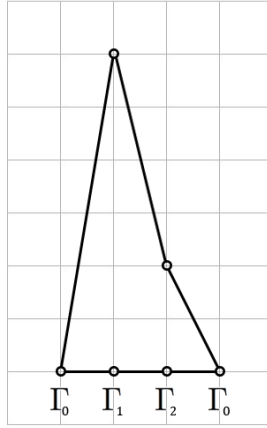


Figure 3: *The graph of the function  $\phi$  considered in Example 4.2.*

**Example 4.2.** Throughout the paper, we will connect the values of  $\phi$  by the unique line between them. This allows us to directly use the Laplacian for subdivisions of the graph. An example of a Laplacian can be found in Figure 3. The graph in question is given in Figure 4. The divisor of the

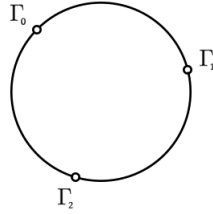


Figure 4: *The graph considered in Example 4.2.*

Laplacian in this figure is

$$\Delta(\phi) = -8(\Gamma_0) + 10(\Gamma_1) - 2(\Gamma_2).$$

We would like to quickly mention a connection between this tropical Jacobian and a well-known theorem on spanning trees in a graph: **Kirchhoff's Theorem**.

**Theorem 4.1.** *Let  $G$  be a finite connected graph. Then the order of the tropical Jacobian of  $G$  is equal to the number of maximal spanning trees in  $G$ .*

**Example 4.3.** Let us take the graph from Example (2.1.3). Then there are three maximal spanning trees, and so the order of tropical Jacobian is three. This of course also means that  $\text{Jac}(G) \simeq \mathbb{Z}/3\mathbb{Z}$ .

*Remark 4.2.* We will later see that the tropical Jacobian is canonically isomorphic to the component group of the Néron model of the Jacobian of  $C$ . Using Kirchhoff's theorem we can say that the order of this component group is then equal to the number of maximal spanning trees.

*Remark 4.3.* Similar to the case of curves, one has multiple ways of constructing a "Tropical Jacobian". In [12] a Tropical Jacobian is constructed using differential forms: one takes the dual  $\Omega(C)^* = \text{Hom}(\Omega(C), \mathbb{R})$  of the space of holomorphic differentials  $\Omega(C)$ , where  $C$  is a tropical curve. By integration, one obtains a lattice  $H_1(\Gamma, \mathbb{Z})$  in this vector space and one then sets

$$J(C) := \Omega(C)^* / H_1(\Gamma, \mathbb{Z}).$$

After choosing a basis of  $\Omega(C)^*$ , one then obtains a noncanonical isomorphism  $J(C) \simeq (\mathbb{R}/\mathbb{Z})^g$ . This Jacobian can then be described entirely in terms of the associated intersection graph, as in [2]. This is already much closer to our approach.

One obvious difference between this approach and our approach is that our Tropical Jacobian is *finite*. As noted in [[1], Remark A.11], we can get somewhat closer by considering the limit over finite extensions  $K' \supset K$  to obtain a  $\mathbb{Q}$ -rational Tropical Jacobian  $J_{\mathbb{Q}}(\Gamma)$  which is noncanonically isomorphic to  $(\mathbb{Q}/\mathbb{Z})^g$ .

Let us describe these phenomena in a particular case: an elliptic curve  $E$  with multiplicative reduction. Over a discretely valued field  $K$  with  $v(\pi) = 1$ , one then obtains an isomorphism with the Tate curve  $E(K) \simeq K^* / \langle q \rangle$  for some  $q$  with positive valuation equal to  $-v(j)$ . One can then define a "naive" tropicalization map

$$\begin{aligned} \text{trop} : (K)^* / \langle q \rangle &\longrightarrow \mathbb{Z} / v(q)\mathbb{Z}, \\ [x] &\longmapsto [v(x)]. \end{aligned}$$

This is easily seen to be well-defined. To study the passage to finite extensions of  $K$ , let us consider the easy example of a ramified extensions of degree  $n$  given by  $K \subset K' := K(\pi^{1/n})$ . We extend the valuation on  $K$  by  $v(\pi^{1/n}) = 1/n$ . As before, one has an isomorphism  $E(K') = (K')^* / \langle q \rangle$ . See [[14], Chapter 5] for this. If we take a similar naive tropicalization as before, one obtains  $\text{trop}(E(K')) = (\frac{1}{n}\mathbb{Z}) / (v(q)\mathbb{Z}) \simeq \mathbb{Z} / (n \cdot v(q)\mathbb{Z})$ . Taking this argument further to an algebraic closure of  $K$ , we then obtain  $\text{trop}(E(\bar{K})) = \mathbb{Q}/\mathbb{Z}$ .

## 4.2 Intersection Theory on $\mathcal{C}$

Here we will start transporting divisors. Let us suppose now that we have a strongly semistable *regular* model  $\mathcal{C}_s$ . As before we will consider its intersection graph  $G$  and the irreducible components  $\{\Gamma_1, \dots, \Gamma_r\}$ . Let  $\text{Div}(C)$  (resp.  $\text{Div}(\mathcal{C})$ ) be the group of Cartier divisors on  $C$  (resp.  $\mathcal{C}$ ). Since both  $C$  and  $\mathcal{C}$  are *regular and integral*, we have by [[9], page 271] that these Cartier divisors correspond to Weil divisors. Similarly, we will let  $\text{Prin}(C)$  (resp.  $\text{Prin}(\mathcal{C})$ ) be the group of principal Cartier divisors on  $C$  (resp.  $\mathcal{C}$ ). Note also that we have that  $\mathcal{C}$  is *normal* (because  $\mathcal{C}_s$  is reduced and  $\mathcal{C}_\eta$  is normal), so we can talk about valuations at codimension 1 primes.

The intersection theory that we now need is described in [[9], page 381] and [[1], page 7]. We will give a very quick summary to set notation. Let  $\text{Div}_s(\mathcal{C})$  be the set of Cartier divisors on  $\mathcal{C}$  with support in  $\mathcal{C}_s$ . These are known as the **vertical divisors**. This group has the  $\Gamma_i$  as a  $\mathbb{Z}$ -basis. (We will later also define the *horizontal divisors*). At any rate, there exists a bilinear map (the intersection map)

$$\text{Div}(\mathcal{C}) \times \text{Div}_s(\mathcal{C}) \longrightarrow \mathbb{Z},$$

which we will write as  $\mathcal{D} \cdot E$  for Cartier divisors  $\mathcal{D}$  and  $E$ , where  $E \subseteq \mathcal{C}_s$ . This can then be computed as

$$\mathcal{D} \cdot E = \deg \mathcal{O}_X(\mathcal{D})|_E.$$

One special case that needs attention is the *self-intersection* of elements of  $\text{Div}_s(\mathcal{C})$ . Suppose we have  $E \subseteq \mathcal{C}_s$ . The number  $E \cdot E$  is called the *self-intersection* of  $E$  and is denoted by  $E^2$ . We then have the following proposition that gives us the self-intersection numbers:

**Proposition 4.1.** Let  $\mathcal{C} \rightarrow S$  be as above. The following properties are then true.

1. For any  $E \in \text{Div}_s(\mathcal{C})$ , we have  $\mathcal{C}_s \cdot E = 0$ .
2. Let  $\Gamma_i$  be the irreducible components of  $\mathcal{C}_s$ . Then for any  $i \leq r$ , we have

$$\Gamma_i^2 = - \sum_{j \neq i} \Gamma_i \cdot \Gamma_j.$$

*Proof.* This is Chapter 9, Proposition 1.21. in [9]. Note that the multiplicities in our case are all 1, so the formula simplifies.  $\square$

*Remark 4.4.* In the semistable case, all intersections will be *transversal*, meaning that

$$\Gamma_i \cdot \Gamma_j = \#\{\text{intersection points of } \Gamma_i \text{ and } \Gamma_j\}.$$

This means that the self-intersection number of any  $\Gamma_i$  is just the total number of intersections with other components.

**Example 4.4.** 1. Let us take  $\mathcal{C} = \text{Proj} R[X, Y, W]/(XY - \pi W^2)$  with affine chart

$$A = R[x, y]/(xy - \pi),$$

where  $x = \frac{X}{W}$  and  $y = \frac{Y}{W}$ . Then  $\Gamma_1 = \overline{(x)}$  and  $\Gamma_2 = \overline{(y)}$ . Then  $\Gamma_1 \cdot \Gamma_2 = 1$  and  $\Gamma_i^2 = -1$ .

2. Let us consider Example 2.1.3 again. We have two components  $\Gamma_1$  and  $\Gamma_2$ . Then  $\Gamma_1 \cdot \Gamma_2 = 3$  and as such we have  $\Gamma_i^2 = -3$ .

Using the intersection theory above, we can now transport our divisors from  $\mathcal{C}$  to  $G$ . We define a homomorphism  $\rho : \text{Div}(\mathcal{C}) \rightarrow \text{Div}(G)$  with

$$\rho(\mathcal{D}) = \sum_{v_i \in G} (\mathcal{D} \cdot \Gamma_i)(v_i).$$

We call this map the *specialization map*.

**Example 4.5.** Suppose we take  $\mathcal{C} = \text{Proj} R[X, Y, W]/(XY - \pi W^2)$  again. Then

$$\rho(\Gamma_1) = \Gamma_1^2(v_1) + (\Gamma_1 \cdot \Gamma_2)(v_2) = -1 \cdot (v_1) + 1 \cdot (v_2).$$

**Example 4.6.** Suppose we now take Example 2.1.3. Then

$$\rho(\Gamma_1) = -3 \cdot (v_1) + 3 \cdot (v_2).$$

Note that this divisor is actually trivial in the Tropical Jacobian. We have that the negative of the characteristic function of the vertex  $v_1$  has divisor equal to  $-3(v_1) + 3(v_2)$ , so  $\rho(\Gamma_1)$  is in the image of  $\Delta$  (the Laplacian). This happens in general: a multiple of the negative of the characteristic function at a vertex  $v_i$  is equal to  $\rho(\Gamma_i)$ , see Lemma 10 in Section 5.2. Thus the image of any vertical divisor in the Tropical Jacobian is *zero*. If we want nontrivial examples of elements of  $\text{Jac}(G)$ , we have to look elsewhere. This is given by the *horizontal divisors*, which we will discuss in the next section.

### 4.3 Transporting divisors from $C$ to $\mathcal{C}$

Now we would like to transport divisors from  $\text{Div}(C)$  to  $\text{Div}(\mathcal{C})$ . Suppose we have any divisor  $D \in \text{Div}(C)$ . We can now take the closure  $\mathcal{D}$  of  $D$  inside  $\mathcal{C}$ . This naturally gives a Cartier divisor of  $\mathcal{C}$ . These are known as the **horizontal divisors**. We will associate a function to the above transportation. Define  $\psi : \text{Div}(C) \rightarrow \text{Div}(\mathcal{C})$  by

$$\psi(D) = \overline{D},$$

where the closure is in  $\mathcal{C}$ .

Let us make the above process a little bit more explicit. Suppose that we have the divisor  $D = P$ , where  $P$  is some point in  $C(K)$ . Then  $P$  specializes to a well-defined point  $r_{\mathcal{C}}(P)$  that lies in the smooth locus of  $\mathcal{C}_s$  ([9], Corollary 9.1.32). *Note that we use regularity here!* See Example (4.8) below for a simple counterexample. At any rate, the point  $P$  reduces to a smooth point and as such it reduces to a unique irreducible component of  $\mathcal{C}_s$ . We will denote this component by  $c(P)$ . We then have  $\mathcal{D} := \overline{\{P\}} = \{P, r_{\mathcal{C}}(P)\}$  and  $\rho(\mathcal{D}) = c(P)$ .

**Example 4.7.** Consider the affine scheme defined by  $R[x, y]/(xy - \pi)$ . It has generic fiber  $K[x, y]/(xy - \pi)$  and special fiber  $k[x, y]/(xy)$ . Consider the point defined by the prime ideal  $\mathfrak{p} = (x - \pi, y - 1)$ . This corresponds to the point on the generic fiber " $(\pi, 1)$ ". There is exactly one maximal ideal lying above  $\mathfrak{p}$ , namely  $\mathfrak{m} = (x - \pi, y - 1, \pi)$  (which corresponds to " $(0, 1)$ " on the special fiber). The closure of the prime ideal  $\mathfrak{p}$  is then  $\{\mathfrak{p}, \mathfrak{m}\}$ . The point  $P$  reduces to a *unique* component, namely the one defined by the prime ideal  $(x)$ .

**Example 4.8.** (*Regularity*) Suppose we now have the affine scheme defined by

$$A := R[x, y]/(xy - \pi^2).$$

It has generic fiber  $K[x, y]/(xy - \pi^2)$  and special fiber  $k[x, y]/(xy)$ . This scheme is however not regular: the tangent space at  $\mathfrak{m} = (x, y, \pi)$  is 3-dimensional, which is strictly higher than the dimension of the ring  $A$  (which is 2).

Consider the prime ideal defined by  $\mathfrak{p} = (x - \pi, y - \pi)$ . This corresponds to the point " $(\pi, \pi)$ " on the generic fiber. There is exactly one maximal ideal above it (this holds for any proper morphism of schemes  $\mathcal{X} \rightarrow S$  where  $S$  is the spectrum of a complete d.v.r.), but there is no *unique* component that it reduces to. Here  $\mathfrak{p} \subseteq \mathfrak{m} = (x, y, \pi)$ , which corresponds to the origin of the coordinate axes. We have that both  $\Gamma_1 := \overline{(x)}$  and  $\Gamma_2 := \overline{(y)}$  contain this point.

*Remark 4.5.* To actually define a reduction for the point in the last example, one can blow-up the point  $\mathfrak{m}$  to obtain a regular model. This works in general, see for instance [9], page 404]. We will see many examples of this phenomenon later on.

**Remark 4.6 (Conventions on divisors).** As noted earlier, since both  $C$  and  $\mathcal{C}$  are *regular and integral*, we have by [9], page 271] that the Cartier divisors correspond to Weil divisors. We will therefore write every Cartier divisor as a *Weil divisor*, i.e. as finite sums of irreducible closed subsets of codimension 1.

Let us give one more notational device regarding principal divisors. Let  $K(\mathcal{C})$  be the function field of  $\mathcal{C}$ . It is equal to the function field of  $C$ . If we have an element  $f \in K(\mathcal{C})$ , we can consider its divisor in both  $C$  and in  $\mathcal{C}$ . To avoid any ambiguity, we will write  $\text{div}(f)$  or  $(f)$  for the divisor in  $\mathcal{C}$  and  $\text{div}_{\eta}(f)$  or  $(f)_{\eta}$  for the divisor in  $C$ .

We will now consider the **principal divisors** of  $C$  and we will see what happens to them under this map  $\psi$ . Unfortunately, if we take a principal divisor  $(f)$  and consider its closure in  $\mathcal{C}$ , then the resulting divisor in  $\text{Div}(\mathcal{C})$  can be nonprincipal! Let us see why this happens.

**Example 4.9.** Suppose we take  $\mathcal{C} = \text{Proj} R[X, Y, W]/(XY - \pi W^2)$  again with affine patch

$$A_1 = R[x, y]/(xy - \pi).$$

It has generic fiber  $K[x, y]/(xy - \pi)$ . Let us take  $x$  in the function field of  $C$ . Then

$$\text{div}_\eta(x) = (0) - (\infty).$$

Note that these points actually don't lie in the affine patch  $A_1$ ; they lie in the other patches determined by  $D^+(X)$  and  $D^+(Y)$  (where the current patch  $A_1$  corresponds to  $D^+(W)$ ).

The function  $x$  can also be considered as an element of the function field of  $\mathcal{C}$  (they are the same after all). To determine this divisor in  $\mathcal{C}$ , we have to know at which codimension 1 primes  $x$  has nonzero valuation. Consider  $\mathfrak{p}_1 = (x, \pi) = (x)$ . The local ring  $A_{1, \mathfrak{p}_1}$  is a discrete valuation ring with generator  $x$ . Thus  $x$  has valuation 1 here. For  $\Gamma_2$  we have the local ring  $A_{1, \mathfrak{p}_2}$  where  $\mathfrak{p}_2 = (y, \pi)$ . The element  $x$  is invertible in this ring, so it has zero valuation. We then in fact have that

$$\text{div}(x) = \overline{\{P\}} - \overline{\{\infty\}} + (\Gamma_1).$$

Note that the closure of  $\text{div}_\eta(x)$  in  $\mathcal{C}$  only contains the first two. In general, for any nonzero element  $f$  of the function field of  $\mathcal{C}$  we can write

$$\text{div}(f) = \overline{\text{div}_\eta(f)} + V,$$

where  $V$  is a vertical divisor (that is defined by the valuations of  $f$  at the vertical divisors).

In fact, if we now have any divisor  $D \in \text{Div}(C)$  of the form  $D = \sum_{P \in C(K)} n_P(P)$ , then we can take the closure  $\mathcal{D}$  of  $D$  in  $\mathcal{C}$  and obtain a divisor there. We have

$$\mathcal{D} = \sum_{P \in C(K)} n_P \overline{(P)} + \sum_i c_i(\Gamma_i),$$

where  $c_i$  is the valuation of  $\mathcal{D}$  at  $\Gamma_i$ .

## 4.4 Jacobians and Néron models

In this section we take the two transporting maps from  $\text{Div}(C)$  to  $\text{Div}(\mathcal{C})$  and from  $\text{Div}(\mathcal{C})$  to  $\text{Div}(G)$  and consider the maps on the Jacobians. There is a description of this map in terms of the Néron model of the Jacobian of  $C$ , which we will present here.

Let  $\text{Div}(C)$  and  $\text{Div}(\mathcal{C})$  be as before. Let  $\text{Div}^0(C)$  be the subgroup of Cartier divisors of degree zero on  $C$ . We further define  $\text{Div}^{(0)}(\mathcal{C})$  to be the subgroup of  $\text{Div}(\mathcal{C})$  consisting of the Cartier divisors such that the restriction of the associated line bundle  $\mathcal{O}_{\mathcal{C}}(\mathcal{D})$  to each irreducible component of  $\mathcal{C}_s$  has degree zero. This last condition can be translated to

$$\deg(\mathcal{O}_{\mathcal{C}}(\mathcal{D})|_{\Gamma_i}) = 0$$

for every  $\Gamma_i$ . Using our specialization map  $\rho$  from before, we can write

$$\text{Div}^{(0)}(\mathcal{C}) = \text{Ker}(\rho).$$

We now let

$$\mathrm{Div}^{(0)}(C) = \{D \in \mathrm{Div}^0(C) : \psi(D) \in \mathrm{Ker}(\rho)\}.$$

As such, it is the inverse image of  $\mathrm{Ker}(\rho)$  under  $\psi$ .

Let us consider the associated Jacobians. To that end, let  $J(C)$  be the Jacobian of  $C$  (that is,  $\mathrm{Div}^0(C)/\mathrm{Prin}(C)$ ) and let  $\mathcal{J}$  be its Néron model (we direct the reader unfamiliar with Néron models to [9],[14] and [4] for introductions to the subject). We let  $\mathcal{J}^0$  be the connected component of the identity in  $\mathcal{J}$ . We denote by  $\Psi = \mathcal{J}_s/\mathcal{J}_s^0$  the group of connected components of the special fiber  $\mathcal{J}_s$  of  $\mathcal{J}$ . This is in fact a finite group that is isomorphic to the Tropical Jacobian we defined earlier! See [[1], page 24] for the details.

**Example 4.10.** Let us take an elliptic curve  $E$  with multiplicative reduction. Its reduction type is thus  $I_n$  and we have that the intersection graph is just a cycle with  $n$  vertices, where  $n = -v(j)$ , where  $j$  is the  $j$ -invariant of  $E$ . We have that  $E$  is canonically isomorphic to its own Jacobian. The Néron model of  $E$  in this case is obtained as follows: one takes the minimal regular model  $\mathcal{C}$ . One then "deletes" all primes that correspond to the intersection points on the special fiber. We then obtain an open subscheme of  $\mathcal{C}$  that we call  $\mathcal{E}$ . We have that  $\mathcal{E}$  is the Néron model of  $E$ . It is an  $\mathrm{Spec}(R)$ -scheme that is *not proper*, but it is a group scheme over  $\mathrm{Spec}(R)$ . Its component group is then equal to  $\mathbb{Z}/n\mathbb{Z}$ . The details can be found in [[9], page 492].

The corresponding analytic version might be useful to have in mind as well. We will follow ([14], Chapter 5). Since  $E$  has multiplicative reduction, we have an analytic isomorphism

$$E(K) \simeq K^*/(q)$$

for some  $q \in K^*$  with  $\mathrm{val}(q) = n$ . We have a natural map

$$i : R^* \longrightarrow K^*/(q),$$

where the image of  $R^*$  in  $E(K)$  is equal to the  $R$ -points of the connected component of the identity  $\mathcal{E}^0$ :

$$i(R^*) = \mathcal{E}^0(R).$$

We then quite easily see that

$$\Psi = (K^*/(q))/(i(R^*)) \simeq \mathbb{Z}/n\mathbb{Z}.$$

Let us now return to the more general case of Jacobians and their Néron models. We can ask for a concrete description of the  $R$ -points of the connected component of the identity and this is given by the following isomorphism:

$$J^0(K) := \mathcal{J}^0(R) \simeq \mathrm{Div}^{(0)}(C)/\mathrm{Prin}^{(0)}(C), \tag{3}$$

where

$$\mathrm{Prin}^{(0)}(C) := \mathrm{Div}^{(0)}(C) \cap \mathrm{Prin}(C).$$

In other words, if we let  $j$  be the injection  $\mathrm{Prin}(C) \longrightarrow \mathrm{Div}(C)$ , then

$$\mathrm{Prin}^{(0)}(C) = (\psi \circ j)^{-1}(\mathrm{Ker}(\rho)).$$

The isomorphism in (3) comes from a theorem by Raynaud, which states that  $\mathcal{J}^0 = \mathrm{Pic}_{C/R}^0$  represents the functor of "isomorphism classes of line bundles whose restriction to each element



of  $\mathcal{C}$  has degree zero". A quick sidenote to clarify this functorial approach: the entities above are considered to be functors from  $(\text{Sch}) \rightarrow (\text{Sets})$ . This identity of functors then means for instance that if we plug in the spectrum of the residue field  $k$  as a scheme, we obtain the identity

$$\mathcal{J}^0(k) = \text{Pic}^0(\mathcal{C}_s)(k). \quad (4)$$

We will study the entity on the right hand side in the next section.

Let us now review some facts about the **torsion** in the Jacobian of a curve. Let  $C$  be a curve of genus  $g$ . We then have the following theorem.

**Theorem 4.2.** *Let  $C$  be a smooth, connected, projective curve over an algebraically closed field  $K$  of genus  $g$ . Let  $n \in \mathbb{Z}$  be non-zero.*

1. *If  $(n, \text{char}(K)) = 1$ , then  $J(C)[n] \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}$ .*
2. *If  $\text{char}(K) = p$ , then there exists an  $0 \leq h \leq g$  such that for any  $n = p^m$  we have  $J(C)[n] \simeq (\mathbb{Z}/n\mathbb{Z})^h$ .*

*Proof.* This can be found in [[9], Theorem 4.38, page 299] or in [[11]]. □

In the rest of the paper, we will not be dealing with the second case of the theorem. This is because in our set-up one runs into separability issues with these torsion points.

## 4.5 Toric structure of $\mathcal{J}^0(k)$

In this section we will further study the  $\mathcal{J}^0(k)$  introduced in the previous section. In fact, we will only study the group  $\text{Pic}^0(X_k)$  for a curve (not necessarily irreducible) over  $k$  (reminder: this is the residue field of  $R$ , which we assume is algebraically closed). We have a natural identification

$$\mathcal{J}^0(k) = \text{Pic}^0(\mathcal{C}_s)(k).$$

from section 4.4 and as such we have a description of  $\mathcal{J}^0(k)$ .

So consider a connected projective curve  $X$  over  $k$  with *smooth* irreducible components  $X_1, \dots, X_r$ . We will follow [[9], Chapter 7, Section 5] with some extra assumptions for the scenario we're interested in. Let us suppose that  $X$  is reduced and that it only has ordinary double points as its singularities (which is the case we're most interested in, the *semistable* case). Let  $X' := \coprod_{1 \leq i \leq r} X_i$  be the *normalization* of  $X$ . We have a surjective integral morphism  $\pi : X' \rightarrow X$ .

**Definition 4.2.**  $\text{Pic}^0(X)$  is the set of isomorphism classes of invertible sheaves  $\mathcal{L}$  such that  $\deg(\mathcal{L}|_{X_i}) = 0$  for every  $1 \leq i \leq r$ .

The structure of  $\text{Pic}^0(X)$  is given by the following theorem. Note that for  $X$  we have a notion of intersection graph. Let  $G$  be this intersection graph.

**Theorem 4.3.** *Let  $X$  be as above (that is, "semistable"). Let  $t = \beta(G)$  be the Betti number of  $G$ . The following properties are then true.*

- a) *The morphism  $\pi$  induces a canonical surjective homomorphism*

$$\pi_{\text{Pic}^0} : \text{Pic}^0(X) \rightarrow \prod_{1 \leq i \leq r} \text{Pic}^0(X_i). \quad (5)$$

b) Let  $L = \text{Ker}(\pi_{\text{Pic}}^0)$ . Then  $L \simeq (k^*)^t$ .

*Proof.* (See [9], page 313, the following is a sketch) Consider the exact sequence of sheaves of abelian groups

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \pi_* \mathcal{O}_{X'}^* \longrightarrow \mathcal{G} \longrightarrow 0, \quad (6)$$

where  $\mathcal{G}$  is a skyscraper sheaf concentrated at the intersection points of the components of  $X$ . Let  $S := \{\text{the intersection points of components of } X\}$ . For any intersection point  $x \in S$  we have the identity on stalks

$$\mathcal{G}_x = (\pi_* \mathcal{O}_{X'}^*)_x / \mathcal{O}_{X,x}^* \simeq k^*$$

(which is Lemma 5.12. on page 309 in [9]). We can take Čech cohomology of sequence (6) to obtain the exact sequence

$$0 \longrightarrow k^* \longrightarrow (k^*)^r \longrightarrow \prod_{x \in S} k^* \longrightarrow \text{Pic}(X) \longrightarrow \text{Pic}(X'), \quad (7)$$

where we used the identification  $H^1(X, \mathcal{O}_X^*) = \text{Pic}(X)$  (which is in [[9], Exercise 5.1.2.7] for instance). The last homomorphism in (7) coincides with the usual homomorphism  $\pi_{\text{Pic}} : \text{Pic}(X) \longrightarrow \text{Pic}(X')$ , which takes  $[\mathcal{L}]$  to  $[\pi_* \mathcal{L}]$ . The theorem now follows from:

1.  $\pi_{\text{Pic}}$  is surjective,
2.  $[\mathcal{L}] \in \text{Pic}^0(X)$  if and only if  $[\pi_* \mathcal{L}] \in \text{Pic}^0(X')$  (this with the previous statement gives (a)),
3. Exactness of the cohomology sequence (7) (which gives (b)).

□

*Remark 4.7.* We will refer to the kernel of  $\pi_{\text{Pic}^0}$  as the **toric part** of  $\mathcal{J}^0$ . It will be denoted by

$$\mathcal{J}_T^0 := \ker(\pi_{\text{Pic}^0}).$$

The elements of  $\mathcal{J}^0$  reducing to nontrivial elements under the map  $\pi_{\text{Pic}^0}$  will be said to belong to the **abelian part** of  $\mathcal{J}^0$ .

## 4.6 Graph cohomology and the toric part of $\mathcal{J}^0(k)$

From Theorem (4.3), we have that the degree zero line bundles consist of an abelian part and a toric part. We will now give a very explicit way to think about these line bundles that come from the toric part in terms of graphs. The reader that is interested in more of this is directed to ([15], Chapter 3). We will mostly follow her presentation of the material, albeit in an algebraic way. So let  $G(V, E)$  be a finite connected graph with vertex set  $V$  and edge set  $E$ . We will review Čech cohomology for this graph with values in an abelian group  $A$  (which for us will be  $k^*$ ).

**Definition 4.3.** A graph  $G(V', E')$  with  $V' \subset V$  and  $E' \subset E$ , where every edge of  $E'$  has source or target in  $V'$  is called a subgraph of  $G(V, E)$ . A subgraph is called complete, if  $E'$  contains all edges of  $E$  with source and target in  $V'$ .

We can now define a topology on  $G$  as follows: the open sets are the complete subgraphs of  $G$ . With this topology we can now define Čech cohomology for graphs. Let  $e$  be any edge of  $G$  and let  $G_e$  be the (complete) subgraph of  $G$  consisting of the edge and the two vertices it joins. We then have the open covering of  $G$

$$\mathfrak{B} = \{G_e : e \in E(G)\}.$$

As with normal Čech cohomology, we now define

$$\check{C}^q(\mathfrak{B}, A) = \prod_{(e_0, \dots, e_q) \in E(G)^{q+1}} A(G_{e_0} \cap \dots \cap G_{e_q})$$

and

$$d_q : \check{C}^q \longrightarrow \check{C}^{q+1}; \alpha \mapsto \left( \prod_{k=0}^{p+1} (-1)^k \alpha_{i_0, \dots, i_{k-1}, i_{k+1}, \dots, i_{q+1}} \right)_{i_0, \dots, i_{p+1}}$$

We then have cohomology groups

$$\check{H}^q(G, A) = \ker d_q / \text{im } d_{q-1},$$

which are trivial for  $q \geq 2$  (since we're working with graphs). Let us describe  $\check{H}^1(G, A)$ . The elements of  $\ker d_1$  are the elements of  $C^1$  that satisfy the cocycle relations

$$\alpha_{e_i, e_j} = \alpha_{e_i, e_k} \cdot \alpha_{e_k, e_j}$$

for three edges sharing a vertex  $v$ . The coboundaries of  $\text{im } d_0$  can then be described by

$$\alpha_{e_i, e_j} = \beta_{e_j} \beta_{e_i}^{-1}$$

for a 0-cocycle  $(\beta_e)_{e \in E}$ . For all the proofs involved, the reader is directed to ([15]).

We will now say that an edge *ends in a vertex*, if said vertex is either target or source of the edge.

**Definition 4.4.** Let  $e$  be an arbitrary edge with target vertex  $v$  and an element  $a \in A$ , we define the weighted cocycle  $\alpha(e, a) = (\alpha_{e_i, e_j})_{e_i, e_j \in E^2}$  by setting

$$\alpha_{e_i, e_j} = \begin{cases} a & \text{if } e_i = e, e_j \neq e \text{ and } e_j \text{ ends in } v, \\ a^{-1} & \text{if } e_j = e, e_i \neq e \text{ and } e_i \text{ ends in } v, \\ 1 & \text{otherwise.} \end{cases}$$

That concludes our short review of graph Čech cohomology on graphs. Let us now return to the scenario of Theorem (4.3). So consider the surjective homomorphism

$$\pi_{\text{Pic}^0} : \text{Pic}^0(X) \longrightarrow \text{Pic}^0(X') = \prod_{1 \leq i \leq r} \text{Pic}^0(X_i)$$

where the  $X_i$  are the irreducible components of  $X$ . This homomorphism can be made quite explicit: one takes a divisor class  $[D]$  on  $X$  and restricts it to all its components:

$$[D] \longmapsto ([D|_{X_i}])_i$$

If we now have a divisor class in the kernel of this map, then this means that for every component  $X_i$ , we can write

$$D|_{X_i} = (f_i)$$

where  $f_i \in k(X_i)$ , the function field of  $X_i$ . Suppose now that we have two intersection points  $x_j$  and  $x_k$  on the same component  $X_i$  of  $X$ . Let the corresponding edges in the intersection graph be given by  $e_j$  and  $e_k$ . We define

$$\alpha_{e_j} = f_i(x_j)$$

and

$$\alpha_{e_j, e_k} = \alpha_{e_j} / \alpha_{e_k}$$

Evaluating this for all edges (or: intersection points) gives a *weighted cocycle* on the intersection graph that corresponds to the element of  $\check{H}^1(G, k^*) = H^1(X, \mathcal{O}_X^*) = \text{Pic}^0(X)$  (the first equality follows from ([15], Proposition 4.2.5)).

## 5 Cartier Divisors, Abelian Covers and Riemann-Hurwitz

In this section we'll study the reduction of divisors a bit more closely by a process which can be described by: "Taking inverse images of Cartier divisors". These inverse images do not always exist, but we'll state a theorem that guarantees the existence of a Cartier divisor on a closed subscheme with the "right" properties. This will also lead to a (non-canonical) definition of the reduction of the divisor of a general element of the function field  $K(\mathcal{C})$ .

We'll follow ([9], 7.1.3, page 260) and add things wherever we need them. So let  $X$  be a scheme and  $D$  be a Cartier divisor on  $X$ .

**Definition 5.1.** The *support* of  $D$ , denoted by  $\text{Supp } D$ , is the set of points  $x \in X$  such that  $D_x \neq 1$ . The set  $\text{Supp } D$  is then a closed subset of  $X$ .

*Remark 5.1.* Recall that the group of Cartier divisors is defined to be  $H^0(X, \mathcal{K}_X^* / \mathcal{O}_X^*)$ , so  $D_x$  is the image of  $D$  in the stalk of the quotient sheaf  $\mathcal{K}_X^* / \mathcal{O}_X^*$  in the point  $x$ .

**Example 5.1.** Let  $\mathcal{C} = \text{Proj } R[X, Y, W] / (XY - \pi W^2)$  with  $\Gamma_1 = \overline{(x)}$  and  $\Gamma_2 = \overline{(y)}$  as before. Consider the Cartier divisor defined by the element  $x$ . As before, we have that

$$\text{div}(x) = \overline{\{P\}} - \overline{\{\infty\}} + (\Gamma_1).$$

We then have

$$\text{Supp}(\text{div}(x)) = \overline{\{P\}} \cup \overline{\{\infty\}} \cup \Gamma_1.$$

Recall that for a locally Noetherian scheme  $X$ , we have a notion of *associated primes*. These are defined by

$$\text{Ass}(\mathcal{O}_X) := \{x \in X : \mathfrak{m}_x \in \text{Ass}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x})\}.$$

**Theorem 5.1.** Let  $X$  be a closed subscheme of a locally Noetherian scheme  $Y$ . Let  $i : X \rightarrow Y$  be the canonical injection.

1. The set  $G_{X/Y}$  of Cartier divisors  $E$  on  $Y$  such that

$$(\text{Supp}(E)) \cap \text{Ass}(\mathcal{O}_X) = \emptyset$$

is a subgroup of  $\text{Div}(Y)$ .

2. There exists a natural homomorphism  $G_{X/Y} \rightarrow \text{Div}(X)$ , denoted by  $E \mapsto E|_X$ , compatible with the homomorphism  $\mathcal{O}_Y \rightarrow i_*\mathcal{O}_X$ . Moreover, we have a canonical isomorphism

$$\mathcal{O}_Y(E)|_X \simeq \mathcal{O}_X(E|_X)$$

and

$$\text{Supp}(E|_X) = \text{Supp}(E) \cap X.$$

If  $E > 0$ , then  $E|_X \geq 0$ . The image of a principal divisor is a principal divisor.

*Proof.* (Sketch) We will outline the construction of the divisor  $E|_X$ . Let  $E$  be represented by  $\{U_i, f_i\}$ , where the  $U_i$  are open in  $Y$ , and  $f_i \in \mathcal{K}_Y^*(U_i)$ . Let

$$\overline{U}_i = X \cap U_i.$$

From the surjective morphism

$$\mathcal{O}_Y \rightarrow \pi_*(\mathcal{O}_X),$$

we obtain a surjective morphism

$$\mathcal{O}_Y(U_i) \rightarrow \mathcal{O}_X(\overline{U}_i),$$

which we denote on the element  $f_i$  as  $\overline{f}_i$ . One can now show that  $\overline{f}_i$  is actually an element of  $\mathcal{K}_X^*(\overline{U}_i)$ , see [[9], page 261] for the details. This then gives a Cartier divisor represented by  $\{(\overline{U}_i, \overline{f}_i)\}_{i \in I}$ .  $\square$

## 5.1 Reducing Cartier divisors for regular semistable models

We will now turn to the case of arithmetic surfaces. Recall that an arithmetic surface is by definition a regular fibered surface. For these surfaces, we have two notions ready: valuations at codimension 1 primes and intersection theory.

We would like to restrict principal divisors to components of the special fiber. So let  $f \in K(\mathcal{D})$  be an element of the function field of  $\mathcal{D}$ . As we saw above, we cannot always restrict the divisor of this element to the special fiber, since we might have that the restricted element is completely contained in the vanishing set of the component (or in other words, there is a nonempty intersection of the divisor of  $f$  with the associated primes of  $\mathcal{D}_s$ ).

We will therefore modify our  $f$  for various primes  $\Gamma \subset \mathcal{D}_s$ . Let  $\Gamma$  correspond to the codimension 1 prime  $\mathfrak{p}$ . The localization  $\mathcal{O}_{\mathcal{D}, \mathfrak{p}}$  is then a discrete valuation ring. The uniformizer  $\pi$  of  $R$  in fact has valuation  $v_{\mathfrak{p}}(\pi) = 1$ , since  $\mathcal{D}_s$  is assumed to be reduced and  $\Gamma$  is contained in the special fiber. Suppose that  $v_{\mathfrak{p}}(f) = k$ .

**Definition 5.2.** The  $\Gamma$ -modified form of  $f$  is defined to be

$$f^\Gamma := \frac{f}{\pi^k}.$$

By definition, we then have  $v_{\mathfrak{p}}(f^\Gamma) = 0$ . If we then consider the natural map

$$\mathcal{O}_{Y, \mathfrak{p}} \rightarrow \mathcal{O}_{Y, \mathfrak{p}} / \mathfrak{p} \mathcal{O}_{Y, \mathfrak{p}},$$

we see that  $f^\Gamma$  naturally gives a nonzero element in the residue field, which we denote by  $\overline{f^\Gamma}$ . Note that the residue field at the prime  $\mathfrak{p}$  is actually the *function field* of the component  $\Gamma$ . We would now like to know the divisor (*in the function field of  $\mathfrak{p}$ !*) of  $f^\Gamma$ . It is quite easy to find this, using Theorem 5.1 and some intersection theory.

**Lemma 9.** *Let  $f$  and  $f^\Gamma$  be as above. We have*

$$\operatorname{div}_\Gamma(\overline{f^\Gamma}) = (\operatorname{div}_Y(f^\Gamma))|_\Gamma.$$

*Proof.* We have that the divisor is represented by  $\{\mathcal{D}, f^\Gamma\}$ , which is then reduced to

$$\{\mathcal{D} \cap \Gamma, \overline{f^\Gamma}\} = \{\Gamma, \overline{f^\Gamma}\}.$$

This is exactly the Cartier divisor  $\operatorname{div}_\Gamma(\overline{f^\Gamma})$ , as desired.  $\square$

Let  $V_f$  and  $V_{f^\Gamma}$  be the vertical divisors of  $f$  and  $f^\Gamma$  respectively. We have that  $V_{f^\Gamma} = V_f - k \cdot \mathcal{D}_s$ . Let  $D^0$  be the closed points of the generic fiber. Recall that we have a natural reduction map

$$r_{\mathcal{D}} : D^0 \longrightarrow \mathcal{D}_s,$$

which associates to every closed point  $x$  in  $D$  the point  $\overline{\{x\}} \cap \mathcal{D}_s$ . We now have

**Proposition 5.1.** Consider the divisor  $\operatorname{div}_\eta(f) = \sum_P n_P(P)$  with corresponding  $\Gamma$ -modified surface divisor

$$\operatorname{div}(f^\Gamma) = \sum_P n_P \overline{\{P\}} + V_{f^\Gamma}.$$

For  $\tilde{x}$  in the nonsingular locus of  $\mathcal{D}_s$ , consider the formal fiber  $D_+(\tilde{x})$ . Then

$$v_{\tilde{x}}(\overline{f^\Gamma}) = \sum_{P \in D_+(\tilde{x})} n_P.$$

For  $\tilde{x}$  an intersection point, say of  $\Gamma$  and  $\Gamma'$ , we have

$$v_{\tilde{x}}(\overline{f^\Gamma}) = v_{\Gamma'}(f^\Gamma).$$

*Proof.* The idea of the proof is to write out the equality in Lemma 9 in terms of valuations.

For  $\tilde{x}$  where  $\overline{f^\Gamma}$  has positive valuation, the valuation can be found by

$$v_{\tilde{x}}(\overline{f^\Gamma}) = \operatorname{length}(\mathcal{O}_{\Gamma, \tilde{x}} / (\overline{f^\Gamma}))$$

(the case with negative valuation is similar). Let  $t$  be a local uniformizer of  $\Gamma$ , so that

$$\mathcal{O}_{\Gamma, \tilde{x}} = \mathcal{O}_{\mathcal{D}, \tilde{x}} / t \mathcal{O}_{\mathcal{D}, \tilde{x}}.$$

We have the equality

$$\mathcal{O}_{\Gamma, \tilde{x}} / (\overline{f^\Gamma}) = \mathcal{O}_{\mathcal{D}, \tilde{x}} / (t \mathcal{O}_{\mathcal{D}, \tilde{x}} + f_{\tilde{x}}^\Gamma \mathcal{O}_{\mathcal{D}, \tilde{x}}).$$

But the length of this last ring is exactly the local intersection number, so that

$$v_{\tilde{x}}(\overline{f^\Gamma}) = (\Gamma \cdot \operatorname{div}(f^\Gamma))_{\tilde{x}}.$$

Writing this condition in terms of the horizontal and the vertical divisors gives us both statements of the proposition.  $\square$

This proposition allows us to calculate the reduced divisor of  $f$  directly in terms of the **horizontal** and the **vertical** divisor of  $f$ .

## 5.2 Vertical Divisors and Intersection Graphs

In the last section we saw that to know the reduced divisors for a given element  $f \in K(\mathcal{D})$ , we have to know the horizontal divisor and the vertical divisor of  $f$ . In this section we will give a way of determining the vertical divisor using the divisors on the intersection graph. To do this, we'll explain in more detail the connection between principal divisors on the intersection graph and vertical divisors.

Suppose we are given an element  $f$  of the function field  $K(\mathcal{D})$ . We have two options: we can consider its divisor in  $D$  and in  $\mathcal{D}$ . The divisor  $\text{div}_\eta(f)$  is well-defined up to a scaling factor of  $K^*$  and the divisor  $\text{div}(f)$  is well-defined up to a scaling factor of  $R^*$ . Namely, for every element of  $K^*$  with nonzero valuation we get a shift in the vertical divisor and for every element of  $R^*$  we obtain the same divisor.

Thus in general it is impossible to reconstruct  $\text{div}(f)$  from just the generic divisor  $\text{div}_\eta(f)$ . If we however take the  $\Gamma$ -modified form  $f^\Gamma$ , we already know that  $v_\Gamma(f^\Gamma) = 0$ . There is then a *unique* solution  $V_f = \sum_i c_i \Gamma_i$  such that  $V_f$  is the vertical divisor corresponding to  $\text{div}_\eta(f)$  with  $c(\Gamma) = 0$ . The good news is that we can explicitly give this vertical divisor in terms of the Laplacian operator.

**Theorem 5.2.** *Let  $\rho(\text{div}_\eta(f))$  be the induced principal divisor of  $f$  on the intersection graph  $G$  of  $\mathcal{D}$ . Write*

$$\Delta(\phi) = \rho(\text{div}(f))$$

*for some  $\phi : \mathbb{Z}^V \rightarrow \mathbb{Z}$ , where  $\Delta$  is the Laplacian operator. Choose  $\phi$  such that  $\phi(\Gamma) = 0$ . Then the unique vertical divisor corresponding to  $\text{div}_\eta(f)$  with  $V_{f^\Gamma}(\Gamma) = 0$  is given by*

$$V_{f^\Gamma} = \sum_i \phi(\Gamma_i) \cdot \Gamma_i. \quad (8)$$

Before we start the proof of this theorem, we'd like to mention an important corollary of this theorem. This corollary says that we can compute "vertical intersections" using the slopes of the function  $\phi$ .

**Corollary 5.1.** Let  $f$  and  $f^\Gamma$  be as before. Recall that we have a natural divisor  $\text{div}_\Gamma(\overline{f^\Gamma})$  on  $\Gamma$ . Let  $\tilde{x}$  be an intersection point of  $\Gamma$  with another component  $\Gamma'$ . Then

$$v_{\tilde{x}}(\text{div}_\Gamma(\overline{f^\Gamma})) = \phi(v') - \phi(v). \quad (9)$$

*Proof.* By proposition 5.1, we find that the valuation of  $\overline{f^\Gamma}$  at  $\tilde{x}$  is equal to the valuation of  $f^\Gamma$  at  $\Gamma'$ . By Theorem 5.2 we see that this is the slope of  $\phi$  in the direction of  $\Gamma$ , as desired.  $\square$

In other words, the slope of  $\phi$  completely determines the multiplicity of  $f$  at the intersection points. We'll use this very often in the rest of the paper.

Let us start by considering the divisors of the simplest functions.

**Lemma 10.** *For any component  $\Gamma_i$  with corresponding vertex  $v_i$ , we have*

$$\Delta(1_{v_i}) = -\rho(\Gamma_i).$$

*Proof.* We calculate both sides. Define

$$B := \{j : \Gamma_i \cap \Gamma_j \neq \emptyset\} \setminus \{i\}.$$

Let  $b = \#B$ . We have

$$\Delta(1_{v_i}) = \left( \sum_{j \in B} -1 \cdot (v_j) \right) + b \cdot v_i.$$

We also have

$$\rho(\Gamma_i) = \left( \sum_{j \in B} (\Gamma_j \cdot \Gamma_i)(v_j) \right) - b \cdot v_i = \left( \sum_{j \neq i: \Gamma_i \cap \Gamma_j} (1)(v_j) \right) - b \cdot v_i.$$

We thus see that

$$\Delta(1_{v_i}) = -\rho(\Gamma_i),$$

as desired. □

*Proof.* (Of Theorem 5.2) Let  $\phi$  be such that

$$\Delta(\phi) = \rho(\operatorname{div}_\eta(f))$$

and  $\phi(\Gamma) = 0$ . We write

$$\phi = \sum_i \phi(\Gamma_i) \cdot 1_{v_i}.$$

Taking the Laplacian operator of  $\phi$  and using Lemma 10, we see that

$$\Delta(\phi) = -\rho\left(\sum_i \phi(\Gamma_i) \cdot (\Gamma_i)\right).$$

Writing

$$\operatorname{div}(f^\Gamma) = \operatorname{div}_\eta(f^\Gamma) + \sum_i c_i(\Gamma_i)$$

and using that  $\rho(\operatorname{div}(f^\Gamma)) = 0$ , we see that

$$\rho\left(\sum_i c_i(\Gamma_i)\right) = \rho\left(\sum_i \phi(\Gamma_i)(\Gamma_i)\right).$$

Since  $c(\Gamma) = 0$  and  $\phi(\Gamma) = 0$ , we must have equality. This gives the theorem. □

**Example 5.2.** Let us consider  $\mathbb{P}^1$  with the function  $f = x(x - \pi)$ . We take the semistable model

$$\operatorname{Proj} R[X, T, W] / (XT - \pi W^2)$$

with open affine

$$\operatorname{Spec} R[x, t] / (xt - \pi).$$

We label the components as  $\Gamma = Z(x)$  and  $\Gamma' = Z(t)$ . We see that

$$\operatorname{div}_\eta(f) = (0) + (\pi) - 2(\infty)$$

and that

$$\rho(\operatorname{div}_\eta(f)) = 2(\Gamma) - 2(\Gamma').$$

The Laplacian thus has slope  $-2$  from  $\Gamma$  to  $\Gamma'$ , as in Figure 5. This means that if we take the  $\Gamma$ -modified form of  $f$ , it will have a pole of order 2 at the intersection point. Furthermore, we see



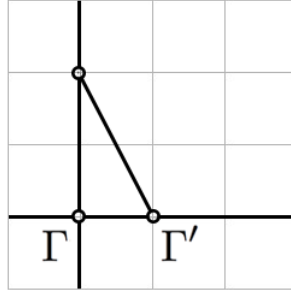


Figure 5: *The Laplacian of the function  $f = x(x - \pi)$ .*

that  $f^\Gamma$  has a zero of order one at  $(0)$  and  $(\pi)$ . This determines  $f^\Gamma$  up to a constant in  $k$ . We can also just calculate the modified form. By writing  $f = x^2(1-t)$ , we easily see that  $v_\Gamma(f) = 2$ . Then the  $\Gamma$ -modified form of  $f$  is equal to

$$f^\Gamma = \frac{1-t}{t^2}.$$

As expected, this has a pole of order 2 at  $t = 0$  and a zero of order 1 at  $(\pi)$ . Furthermore, it has a zero of order one at  $t = \infty$ , which corresponds to the point  $(0)$ , as expected.

Let us now determine the  $\Gamma'$ -modified form of  $f$ . We have that the Laplacian has slope 2 and thus that  $f^{\Gamma'}$  has a zero of order two at the intersection point. Furthermore, we see that  $f^{\Gamma'}$  has a pole of order two at  $(\infty)$ . We calculate the  $\Gamma'$ -form. Since  $v_{\Gamma'}(f) = 0$ , we can just substitute  $t = 0$ . We then obtain

$$f^{\Gamma'} = \bar{x}^2$$

which has a zero of order 2 at  $x = 0$  (which corresponds to the intersection point) and a pole of order two at  $\infty$ .

### 5.3 Abelian coverings

We quickly recall some Kummer Theory. Since we'll only deal with extensions of prime degree, we'll stick to that case. Suppose we have a finite Galois extension  $K \subseteq L$  of degree  $q$  (with  $q$  prime) coprime to the characteristic  $p$  of  $K$ . We will also suppose that  $K$  contains a primitive  $q$ -th root of unity  $\zeta_q$ . If  $q$  is not coprime to the characteristic, one has to consider so-called Artin-Schreier type extensions. We will not pursue this path here. For our case, the abelian extension takes a very simple form.

**Proposition 5.2.** (Simplified Kummer Theory) Let  $K \subseteq L$  be a finite Galois extension of degree  $q$  with  $q$  coprime to the characteristic  $p$  of  $K$ . Suppose that  $K$  contains a primitive  $q$ -th root of unity. We then have

$$L = K[X]/(X^q - f)$$

for some  $f \in K$ .

We now return to the Dedekind case, where we have a finite extension of Dedekind domains

$$A \longrightarrow A'$$

with quotient fields  $K \subset L$  as before. We can consider the subalgebras

$$A \subseteq A[X]/(X^q - f) \subseteq A',$$

where  $f$  is as in Proposition 5.2. We do not always have equality, as the extension given by a certain  $f$  might be *nonnormal*. We will now look at the local case. So we assume that  $A$  is local and Dedekind, meaning that it is a discrete valuation ring. We take  $\mathfrak{p}$  to be the maximal ideal and  $t$  to be the uniformizer of  $A$ . Consider any  $f$  from Proposition 5.2. We can then write  $f = t^n u$  where  $u$  is a unit and  $n \geq 0$ .

**Proposition 5.3.** Let  $A \subseteq A[X]/(X^q - f) \subseteq A'$  be as above with  $f = t^n u$ . We then have that the extension is unramified if and only if  $n \equiv 0 \pmod q$ .

*Proof.* One can consider the Newton polygon of  $X^q - t^n u$ , which is given by a single line segment of slope  $-\frac{n}{q}$ . We then have that the extension is unramified  $\iff$  The slope is integral  $\iff n \equiv 0 \pmod q$ . This gives the proposition.  $\square$

**Corollary 5.2.** For any abelian extension  $A \subseteq A'$  of Dedekind domains with corresponding extension of fraction fields  $K \subseteq L = K[x]/(X^q - f)$ , we have that the extension is unramified above  $\mathfrak{p}$  if and only if  $v_{\mathfrak{p}}(f) \equiv 0 \pmod q$ .

### 5.4 Semistable Riemann-Hurwitz for abelian coverings

In this section we will state and prove a *semistable version* of the well-known Riemann-Hurwitz formula for abelian coverings. The need for such a formula arises as follows: one takes an abelian covering  $C \longrightarrow D$  of some curve  $D$  with strongly semistable model  $\mathcal{D}$ . One would then like to know the genus of a component  $\Gamma_C$  lying above a component  $\Gamma_D$ . In order to do this, one needs to know the ramification of the map  $\Gamma_C \longrightarrow \Gamma_D$ . This can be given in terms of the vertical divisors discussed earlier. The reason that we work only with abelian coverings is because in this case we

have an easy criterion of determining whether the extension is ramified or not, namely Corollary 5.2.

So suppose we have a strongly semistable covering  $\phi : \mathcal{C} \rightarrow \mathcal{D}$  of degree  $q$ . Let us fix a component  $\Gamma_D$  in the special fiber of  $\mathcal{D}$ . Then on an open affine  $\text{Spec}(A)$ ,  $\Gamma_D$  corresponds to a prime ideal of codimension 1:  $\mathfrak{p}$ . For any prime dividing  $\mathfrak{q}$ , we have a finite extension of residue fields

$$k(\mathfrak{p}) \rightarrow k(\mathfrak{q}).$$

We let  $\Gamma_C$  be the component corresponding to  $\mathfrak{q}$ . The extension of residue fields then corresponds to a morphism of curves over  $k$

$$\Gamma_C \rightarrow \Gamma_D.$$

The degree of this morphism of curves is equal to  $f_{\mathfrak{q}/\mathfrak{p}}$ . It is an abelian covering of  $\Gamma_D$  with Galois group  $D_{\mathfrak{q}/\mathfrak{p}}$ . See Proposition 3.7 for this. By our assumption on the degree of the extension, we have only two options:

$$f_{\mathfrak{p}} = q, \tag{10}$$

$$f_{\mathfrak{p}} = 1. \tag{11}$$

In the first case, we have that there exists only one prime above  $\mathfrak{p}$ , with a Galois extension of degree  $f_{\mathfrak{p}}$ . In the second case, we have that  $\mathfrak{p}$  is *completely split*, in the sense that there are  $q$  primes dividing  $\mathfrak{p}$ .

We now consider the extension

$$A_{\mathfrak{p}} \rightarrow A_{\mathfrak{p}}[X]/(X^q - f),$$

where  $f$  is some element giving the abelian extension. In fact, since  $\phi$  is a morphism of strongly semistable coverings (and we thus have no ramification), we can assume that  $v_{\mathfrak{p}}(f) = 0$ . Let  $\bar{k} := k(\mathfrak{p})$ . We can then consider the morphism on residue fields

$$\bar{k} \rightarrow \bar{k}[X]/(X^q - \bar{f}).$$

In order to determine in what case ( $f_{\mathfrak{p}} = q$  or  $f_{\mathfrak{p}} = 1$ ) we are in, we have to determine whether  $X^q - \bar{f}$  is irreducible. We would like to do this in terms of divisors. There are two cases here that we will have to consider: the genus 0 case and the positive genus case.

**Proposition 5.4.** Suppose that  $g(\Gamma_D) = 0$ . Then  $X^q - \bar{f}$  is reducible if and only if  $\text{div}(\bar{f}) \equiv 0 \pmod{q}$ .

*Proof.* If  $X^q - \bar{f}$  is reducible, then we can find a solution  $x_0 \in \bar{k}$  such that  $x_0^q = \bar{f}$ . We then have  $\text{div}(x_0^q) = q \cdot \text{div}(x_0) = \text{div}(\bar{f})$ . Conversely, if  $\text{div}(\bar{f}) = qD$  for some divisor  $D$  of degree 0, then we can write

$$D = \text{div}(f')$$

for some element  $f'$ . We then have that a scaled version of  $f'$  solves the equation  $X^q = f$ .  $\square$

**Proposition 5.5.** Suppose that  $g(\Gamma_D) > 0$ . Then  $X^q - \bar{f}$  is reducible if and only if  $\bar{f}$  is a constant.

*Proof.* If there exists a point  $\bar{P}$  with  $v_{\bar{P}}(\bar{f}) \not\equiv 0 \pmod{q}$ , then the equation  $X^q - \bar{f}$  is irreducible at  $\bar{P}$ . If  $v_{\bar{P}}(\bar{f}) \equiv 0 \pmod{q}$  for every  $\bar{P}$ , then we either have a  $q$  torsion point of the Jacobian, or a constant. The  $q$ -torsion case corresponds to irreducibility of the above equation.  $\square$

#### 5.4.1 Semistable Riemann-Hurwitz

In this section we will prove the semistable Riemann-Hurwitz formula. The main idea is as follows: For every finite separable covering of smooth irreducible curves  $X_1 \rightarrow X_2$  we have a Riemann-Hurwitz formula. In our case, we have two of these coverings in two different worlds: on the one hand we have a covering on the generic fiber:

$$\mathcal{C}_\eta \rightarrow \mathcal{D}_\eta$$

which is governed by the "generic" ramification points. On the other hand, for any covering of components in the special fiber

$$\Gamma_C \rightarrow \Gamma_D$$

we also have a Riemann-Hurwitz formula which depends on the ramification points in the special fiber. These two types of ramification points are *linked* by Propositions 5.7 and 5.8. In other words, what we call a "semistable" Riemann-Hurwitz formula is a way of linking the ramification points on the generic fiber to the ramification points on the components in the special fiber.

So let us start with some preliminary considerations. Recall that in the previous section we considered the *reducible* case, which corresponds to  $D_{\mathfrak{q}/\mathfrak{p}} = (1)$ . We now turn to the  $D_{\mathfrak{q}/\mathfrak{p}} = \mathbb{Z}/q\mathbb{Z}$  case. We then have a single extension of function fields of degree  $q$ . We would now like to know the genus of  $\Gamma_C$ . We will use the original Riemann-Hurwitz formula in the Galois case.

**Proposition 5.6.** Let  $\phi_{\Gamma_C} : \Gamma_C \rightarrow \Gamma_D$  be a Galois morphism of degree  $q$  between two smooth irreducible curves. Then

$$2g_C - 2 = \deg(\phi)(2g_D - 2) + (q - 1)\#\mathcal{R},$$

where

$$\mathcal{R} = \{\overline{P} \in \Gamma_D(k) : \phi \text{ is ramified above } \overline{P}\}.$$

*Proof.* In the Galois prime case we have that every ramified point has the same ramification index:  $q$ . The formula above then follows from the usual Riemann-Hurwitz formula.  $\square$

Thus in order to know the genus of the curve  $\Gamma_C$ , we have to know the ramification locus of  $\phi$ . We will give a criterion for  $\phi_{\Gamma_C}$  to be ramified at the intersection points first.

**Proposition 5.7 (Criterion for ramification at intersection points).** Let  $f$  be a principal divisor on  $K(D)$ . Let  $\phi$  be the associated Laplacian. Let  $f^{\Gamma_D}$  be the adjusted function for the component  $\Gamma_D$ . Suppose that the induced map  $\phi_{\Gamma_C} : \Gamma_C \rightarrow \Gamma_D$  is of degree  $q$ . Let  $x$  be any intersection point of  $\Gamma_D$  with another component. Let  $\delta_\phi(x)$  be the slope of  $\phi$  in the direction of  $x$  at  $\Gamma_D$ . We then have:

$$\phi_{\Gamma_C} \text{ is unramified above } x \text{ if and only if } \delta_\phi(x) = 0 \pmod{q}.$$

*Proof.* This follows from Corollary 5.1 and Proposition 5.3.  $\square$

We can state this proposition a little bit differently: the ramification of the morphism  $\phi_{\Gamma_C}$  can be found by identifying the part of the intersection graph  $\mathcal{G}(D)$  where the Laplacian corresponding to  $f$  is nontrivial.

Now we consider the other "ordinary" points. We similarly have a criterion:

**Proposition 5.8 (Criterion for ramification at ordinary points).** Let  $f$ ,  $\Gamma_D$  and  $f^{\Gamma_D}$  be as before. Let  $\bar{P}$  be a nonintersection point of  $\Gamma_D$ . Let

$$\mathcal{R}_{\bar{P}} = \{P \in K(D) : \text{red}(P) = \bar{P}\}.$$

Define

$$r_P = \sum_{P \in \mathcal{R}_{\bar{P}}} v_P(f).$$

Then

$$r_P = v_{\bar{P}}(\overline{f^{\Gamma_D}}).$$

*Proof.* This is just restating Proposition 5.1. □

**Corollary 5.3.**  $\phi_{\Gamma_C}$  is unramified at  $\bar{P}$  if and only if  $r_P = 0 \pmod q$ .

We can thus quite easily determine the genus of the component  $\Gamma_C$  using *only information about the divisor of  $f$  and the reduction of the points in the divisor*.

**Example 5.3.** Suppose that we are given an elliptic curve  $E$  over  $K$  with multiplicative reduction, with a 3-torsion point  $P$  reducing to the singular point. An explicit family of these curves can be found in [20]. This point  $P$  then gives a point of order three in the component group of the Néron model of  $E$ . By definition, there exists a function  $f$  such that

$$(f) = 3(P) - 3(\infty).$$

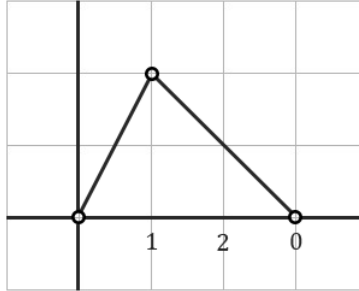


Figure 6: *The Laplacian of  $f$ .*

Subdividing the reduction graph  $\mathcal{G}(E)$  into three equidistant parts, we see that  $P$  must reduce to one third of the length of  $\mathcal{G}(E)$ , whose component we denote by  $\Gamma_1$ . We thus have the Laplacian

$$\rho(\text{div}_\eta(f)) = 3(\Gamma_1) - 3(\Gamma_0).$$

We take the solution  $\phi$  with

$$\begin{aligned} \phi(0) &= 0, \\ \phi(1) &= 2, \\ \phi(2) &= 1, \end{aligned}$$

which has slope 2 between  $\Gamma_0$  and  $\Gamma_1$  on the left side and slope 1 between  $\Gamma_1$  and  $\Gamma_0$  on the right side, as in Figure 6.

If we consider the extension

$$z^3 = f,$$

then this gives a morphism  $E' \rightarrow E$ , which ramifies twice at every component (namely at the intersection points), since the slope is not divisible by 3. The reduction graph is thus the same and  $E'$  is an elliptic curve with multiplicative reduction. This was to be expected from an isogeny of two elliptic curves where one has bad reduction, see [[13], Chapter VII, Corollary 7.2.]. The covering of graphs can be found in Figure 7.

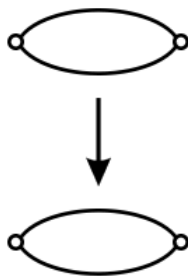


Figure 7: *The covering of graphs in Example 5.3.*

**Example 5.4.** Suppose we take Example 2.13 again, with the banana graph of genus 2, as in Figure 8. The corresponding equation is

$$y^2 = x(x - \pi)(x + 1)(x + 1 - \pi)(x + 2)(x + 2 - \pi).$$

We label the components by  $\Gamma$  and  $\Gamma'$ . There is a natural 3-torsion point  $D'$  on this graph, namely



Figure 8: *The intersection graph of the genus 2 curve in Example 5.4.*

$$D' = (\Gamma) - (\Gamma').$$

Suppose we have a divisor  $D$  of order 3 in the Jacobian of  $C$  such that  $\rho(D) = D'$ . For some function  $f$ , we have that

$$3D = \text{div}_\eta(f).$$

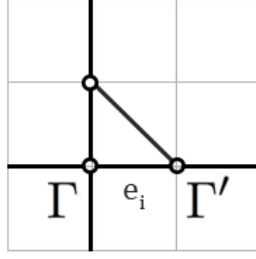


Figure 9: The Laplacian function  $\phi$  of  $f$ , as in Example 5.4. The  $e_i$  denote the three edges between the two vertices.

Then  $\rho(\text{div}_\eta(f)) = 3D'$  and the corresponding Laplacian function up to scaling is just the indicator function of  $\Gamma$ , as in Figure 9. We thus see that the morphism on the components is ramified at every vertex, with the vertices  $\Gamma_0$  and  $\Gamma_1$  having three ramification points and the ones elsewhere having only 2. Using the Riemann-Hurwitz formula, we see that the primes dividing  $\Gamma_0$  and  $\Gamma_1$  have genus 1 and the edges connecting them have genus 0. The graph thus consists of two vertices with weights 1 and three edges connecting them. This gives a genus 4 graph, as expected. The covering of graphs can be found in Figure 10.

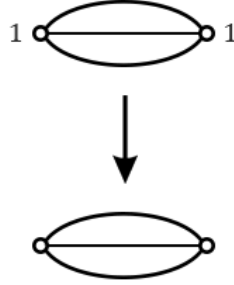


Figure 10: The covering of graphs in Example 5.4.

**Example 5.5.** Examples involving ramified coverings of curves will be given in Sections 7 and 8. The corresponding situation is not a lot harder: one determines  $r_{\mathcal{D}}(P)$  (the reduction of  $P$ ) for every ramification point  $P$  and then uses Proposition 5.8.

## 6 Unramified morphisms of curves and their intersection graphs

We saw in the last section that it is possible to determine explicitly for a component  $\Gamma_D$  what its "dividing primes"  $\Gamma_C$  will be. We saw that it was then possible to give the genus of any such component  $\Gamma_C$ . This solves the reduction type problem for most abelian coverings. However, there is one problem that we have circumvented thus far: nontrivial coverings of graphs. In this section we will study these and call them "*Étale Morphisms of Graphs*".

So suppose we have a Galois morphism of strongly semistable models  $\phi : \mathcal{C} \rightarrow \mathcal{D}$ . Recall that we have that  $\mathcal{C}/G = \mathcal{D}$ . In the last few sections, we studied this action on the codimension 1 primes of the special fiber. We will now more closely study the intersection points. The questions that we will want to answer here are: Suppose we have two components  $\Gamma_D$  and  $\Gamma'_D$  in  $\mathcal{D}_s$  with intersection points  $x_1, x_2, \dots, x_r$ . How can we predict the intersection points lying above the  $x_i$ ? And to what components do these new intersection points belong?

We will start with some preliminary considerations. Suppose we have an intersection point  $x$  lying on  $\Gamma_D$  and  $\Gamma_{D'}$ . This intersection point corresponds to an edge on the intersection graph  $\mathcal{G}(\mathcal{D})$ . Our first question is: what is  $g_x$ ? That is, how many points are there above  $x$ ? To do that, we consider the following alternation. We *subdivide* our intersection graph. Meaning, if we write the local ring at intersection point as

$$R[[x, y]]/(xy - \pi^n),$$

then first we check if  $n \equiv 0 \pmod{2}$ . If not, we make a ramified extension of degree 2. We can then find a component  $\Gamma''_D$  lying between  $\Gamma_D$  and  $\Gamma'_{D'}$ . This means that it intersects  $\Gamma_D$  and  $\Gamma'_{D'}$  exactly once.

We then do the same thing as before: we consider the ramification of  $f$  (the function defining our abelian extension) at the component  $\Gamma''_D$ . This will give us the  $g_x$ . In practice, we never have to calculate  $\Gamma''_D$ : one finds the Laplacian of  $f$  and then determines the ramification abstractly at the point  $\Gamma''_D$ . This will give the order of the decomposition group  $|D_e|$ .

Knowing the order of the decomposition group is not enough however to reconstruct the intersection graph  $\mathcal{G}(\mathcal{C})$ . We will consider the following extreme case, which will highlight this problem:

- $|D_{e'}| = (1)$  for every edge  $e'$  in  $\mathcal{G}(\mathcal{C})$ .

This will lead to the definition of *étale morphisms of graphs*, and later to the definition of *completely decomposable morphisms of graphs*.

### 6.1 Étale morphisms of graphs

Let us consider the following problem. Suppose we have a disjointly branched covering  $\mathcal{C} \rightarrow \mathcal{D}$  with Galois group  $\mathbb{Z}/q\mathbb{Z}$ . Suppose that for every edge  $e'$  of  $\mathcal{G}(\mathcal{C})$  dividing an edge  $e$  in  $\mathcal{G}(\mathcal{D})$ , we have that  $D_{e'} = (1)$ . In other words, we know for every edge on the intersection graph  $\mathcal{G}(\mathcal{D})$  that there are exactly  $q$  edges lying above them. If in addition the morphisms on components

$$\Gamma_C \rightarrow \Gamma_D$$

are all unramified, we will say that the induced morphism of graphs

$$\mathcal{G}(\mathcal{C}) \rightarrow \mathcal{G}(\mathcal{D})$$

is "étale".



**Definition 6.1.** Suppose we have a disjointly branched, abelian Galois morphism  $\phi : \mathcal{C} \rightarrow \mathcal{D}$  of degree  $q$  such that for every edge  $x$  in  $\mathcal{G}(\mathcal{D})$  there exist exactly  $q$  edges dividing  $x$ . Suppose in addition that the morphisms

$$\Gamma_C \rightarrow \Gamma_D$$

on components are all unramified. Then this morphism  $\phi$  with the corresponding morphism on the reduction graphs is then referred to as an **étale morphism of graphs**.

Let us now see why this terminology of "étale" abelian morphisms of graphs is appropriate.

**Lemma 11.** *Let  $\phi : \mathcal{C} \rightarrow \mathcal{D}$  be disjointly branched of degree  $q$ . Then  $\phi_{\mathcal{G}}$  is an étale morphism of graphs if and only if  $\phi$  is étale on the intersection graph  $\mathcal{G}(\mathcal{C})$  and on the components  $\Gamma_C \rightarrow \Gamma_D$ .*

*Proof.* First note that  $\phi$  is always étale at primes corresponding to components. Furthermore, we see that  $\phi$  is étale at an intersection point if and only if there are  $q$  pre-images. These two conditions quickly give the lemma.  $\square$

**Lemma 12.** *Let  $\phi : \mathcal{C} \rightarrow \mathcal{D}$  be a disjointly branched of degree  $q$ . Suppose that  $\phi_{\mathcal{G}}$  is étale. Then  $\phi_{\eta}$  is unramified.*

*Proof.* Suppose for a contradiction that  $\phi_{\eta}$  is ramified. Then there exists a point of  $D$  where it ramifies:  $P$ . The closure of  $P$  in  $\mathcal{D}$  is by assumption disjoint from any other ramification points. Let  $\Gamma$  be the component that  $P$  reduces to, with corresponding prime  $\mathfrak{p}$ . For a prime  $\mathfrak{q}$  dividing  $\mathfrak{p}$ , we have that  $\overline{P}$  will be a ramification point of  $\overline{\phi}$ . Indeed, if  $n = v_P(f)$  for an  $f$  defining the extension, then by the disjointness of the closures of the ramification points we have

$$v_P(f) = v_{\overline{P}}(\overline{f}) \pmod{q}.$$

This contradicts the fact that the morphism  $\Gamma_C \rightarrow \Gamma_D$  is unramified, as desired. Note that in the last equation we need to consider this equality mod  $q$ , because there might be points in the support of  $(f)_{\eta}$  that don't ramify and reduce to  $\overline{P}$ . For these points we have  $v_P(f) = 0 \pmod{q}$ , so it doesn't affect the local valuation.  $\square$

*Remark 6.1.* Note that the converse is definitely not true, since we can have an unramified morphism  $\phi : C \rightarrow D$  with edges having  $D_e = \mathbb{Z}/q\mathbb{Z}$ . This happens for instance if we take an elliptic curve  $E$  with multiplicative reduction with a  $q$ -torsion point that reduces to the singular point. The corresponding extension is unramified and yields the same reduction graph as  $E$ . See Example 5.3 for instance.

We thus see that these étale morphisms of graphs correspond to a certain class of abelian unramified morphisms of curves. Recall that the abelian unramified coverings of a curve  $D$  over a field are classified by the isomorphism:

$$\mathrm{Hom}(\pi_1(D), \mathbb{Z}/n\mathbb{Z}) \simeq J(D)[n].$$

for  $n$  prime to the characteristic of the ground field  $K$ . In fact, the way one creates all unramified abelian coverings of a curve (and in fact of any nonsingular variety over a field containing  $\mu_n$ ) is as follows. One starts with an  $n$ -torsion point  $P$  in the Jacobian. This means that there is a function  $f$  such that

$$\mathrm{div}(f) = nP.$$

The corresponding extension of function fields given by  $z^n = f$  then gives an unramified covering of the curve  $D$ .

Let us now found out what unramified abelian extensions correspond to étale morphisms of graphs.

**Proposition 6.1.** Let  $P \in J(D)[q]$  be a  $q$ -torsion point giving rise to an unramified abelian morphism of degree  $q$ :

$$\phi : C \longrightarrow D.$$

Then the induced morphism

$$\phi_{\mathcal{G}} : \mathcal{G}(\mathcal{C}) \longrightarrow \mathcal{G}(\mathcal{D})$$

is an étale morphism of graphs if and only if  $D \in \mathcal{J}^0(R)[q]$ .

*Proof.* If  $P \in \mathcal{J}^0(R)[q]$ , then we see that the Laplacian corresponding to  $qP$  is zero everywhere. For every edge in  $\mathcal{G}(\mathcal{D})$ , we then have  $q$  primes lying above it, so we have an unramified morphism of graphs.

Conversely, suppose that we have an unramified morphism of graphs. Then for every edge we have that the Laplacian has slope divisible by  $q$ . We can then quite easily find a new function  $\phi'$  such that  $q\Delta(\phi') = \Delta(\phi)$ . But then  $\rho(P) = \Delta(\phi')$  and so the class of  $P$  is in the identity component of  $\mathcal{J}(D)$ , as desired.  $\square$

**Example 6.1.** Suppose we take a genus 2 curve  $D$  with reduction graph consisting of two vertices and two edges. We label the 2 corresponding components by  $\Gamma_0$  and  $\Gamma_1$ . One of them must have genus one, so let that component be  $\Gamma_0$ . We now take a 3-torsion point in the Jacobian of  $D$  that

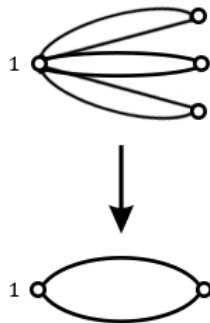


Figure 11: *The covering in Example 6.1.*

reduces entirely to  $\Gamma_0$ . That is, we take a 3-torsion point of the corresponding genus 1 curve. If we consider the extension defined by that 3-torsion point, we obtain the intersection graph consisting of 4 vertices, 3 lying above  $\Gamma_1$  and 1 above  $\Gamma_0$  with 2 edges between each component  $\Gamma'_1$  and  $\Gamma'_0$  (so 6 in total). The resulting covering of intersection graphs is in Figure 11. Note that the component  $\Gamma'_0$  again has genus 1, since it is given as an unramified covering of a genus 1 curve. The Betti number of the graph is 3 and the total genus is  $3 + 1 = 4$ , as expected.

Note that the covering of graphs in this case is *unramified*: for every edge there are exactly three pre-images. The Galois action then permutes these edges accordingly.

## 6.2 Completely decomposable morphisms of graphs

We will now continue our study of unramified morphisms of graphs. As we saw in the last section, they arise from  $q$ -torsion points in the identity component  $\mathcal{J}^0$  of the Jacobian  $J(D)$ . For these morphisms we know the decomposition groups of the edges:  $D_e = (1)$  for every edge  $e$  in  $\mathcal{G}(\mathcal{C})$ . This does not fix the order of the decomposition groups of the vertices however. In this section we will study the completely reducible case, where  $D_v = (1)$  for every vertex.

**Definition 6.2.** Suppose that we are given an unramified Galois cover of graphs  $\phi_\Sigma : \Sigma_1 \rightarrow \Sigma_2$  with Galois group  $\mathbb{Z}/q\mathbb{Z}$ . Suppose that  $D_v = (1)$  for every vertex. Then  $\phi_\Sigma$  is referred to as a **completely decomposable morphism of graphs**.

**Example 6.2.** Suppose we take an elliptic curve  $E$  with multiplicative reduction. We can write the reduction graph as two vertices with two edges between them. We now take a 2-torsion point  $P$  reducing to a *nonsingular* point. This means that it is a 2-torsion element in the generic fibre of the identity component of the Néron model. The corresponding extension is completely reducible everywhere (we will in fact write down the equations explicitly soon, where it will be clear why this is true). We thus obtain four vertices with four edges between them. The graph has to be connected, so there is only option. Note that for any component  $\Gamma$  of  $\mathcal{E}$ , the two primes lying above  $\Gamma$  do not intersect. There is thus only the *trampoline* option, given by connecting the vertices of  $\mathcal{G}(E')$  transversally. We thus also find that  $E'$  has multiplicative reduction. The covering of graphs can be found in Figure 12.

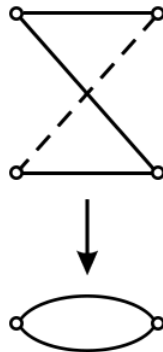


Figure 12: *The covering in Example 6.2.*

**Proposition 6.2.** Let  $P \in J(D)[q]$  be a  $q$ -torsion point giving rise to an unramified abelian morphism of degree  $q$ :

$$\phi : C \rightarrow D.$$

Then  $\phi_{\mathcal{G}} : \mathcal{G}(\mathcal{C}) \rightarrow \mathcal{G}(\mathcal{D})$  is a completely decomposable morphism of graphs if and only if

$$P \in \mathcal{J}_T^0[q].$$

*Proof.* Suppose that

$$\mathcal{G}(\mathcal{C}) \rightarrow \mathcal{G}(\mathcal{D})$$

is completely decomposable. Then for every vertex  $v$  in  $\mathcal{G}(\mathcal{D})$ , we have that  $g_v = q$ . Using Propositions 5.4 and 5.5, we see that the reduced divisor  $(f)|_v$  is trivial in every

$$\text{Pic}^0(\Gamma_i),$$

so that  $P$  is in fact a toric divisor by Theorem 4.3.

Now suppose that  $P$  is a toric divisor. We then see that the pulled back divisors are trivial for every component. Using Propositions 5.4 and 5.5 again, we see that the corresponding extensions have  $g_v = q$ , as desired.  $\square$

### 6.2.1 Explicit computations for completely decomposable morphisms

We saw in the previous section that these *morphisms of graphs* are enough to distinguish between  $\mathcal{J}$ ,  $\mathcal{J}^0$  and  $\mathcal{J}_T^0$ . We would now like to distinguish between the different toric divisors and their corresponding extensions. To do this, we will take a strong hint from graph cohomology and apply it to our setting. We will give a method of determining what kind of reduction we obtain in this case from the given equations. This boils down to finding a suitable étale algebra and then determining the maximal and codimension 1 prime ideals.

So suppose we have a **toric**  $q$ -torsion point  $[P]$  in  $\mathcal{J}^0(K) = \text{Div}^{(0)}(D)/\text{Prin}^{(0)}(D)$  with representative  $P \in \text{Div}^{(0)}(D)$ . If we restrict  $P$  to a component  $\Gamma$ , then we can write

$$P|_{\Gamma} = \text{div}(\overline{h_{\Gamma}})$$

for some  $\overline{h_{\Gamma}}$  in the function field  $k(\Gamma)$  of the component  $(\Gamma)$ , because  $P$  is trivial. A local lift of  $\overline{h_{\Gamma}}$  to  $\mathcal{D}$  will be denoted by  $h_{\Gamma}$ . Since  $P$  is a  $q$ -torsion point, we have that

$$q \cdot P = \text{div}_{\eta}(f)$$

for some  $f \in \text{Prin}^{(0)}(D)$ . If we restrict this equality, then locally we have that

$$(\overline{f})_{\Gamma} = (\overline{h_{\Gamma}})^q,$$

meaning that  $f$  is locally a  $q$ -th power. We *assume* here that we have scaled  $h_{\Gamma}$  such that

$$\overline{f}(x) = (\overline{h_{\Gamma}}(x))^q.$$

Returning to the equation  $\overline{f}_{\Gamma} = \overline{h_{\Gamma}}^q$ , there are exactly  $q$  functions that have this property, so we will introduce notation for them. Let  $\zeta := \zeta_q$ . We define

$$h_{\Gamma,i} := \zeta^i \cdot h_{\Gamma}.$$

for every component  $\Gamma$  in  $\mathcal{D}_s$ . As we will see, these functions correspond to the components that lie above  $\Gamma$ .

Now let  $x$  be an intersection point of two components  $\Gamma$  and  $\Gamma'$  in  $\mathcal{D}_s$ . By assumption on  $f$ , we find that there is a well-defined value

$$\overline{f}(x) \in k^*.$$

We now define for any  $x$  an intersection point the following set:

$$\mathcal{S}_x = \{\alpha \in k^* : \alpha^q = \overline{f}(x)\}.$$

We will see that these elements correspond exactly to the  $q$  intersection points lying above  $x$ .

Now let us review the situation we are in. Let us consider  $f$  as an element of the function field of  $K(\mathcal{D})$ . For every  $x \in \mathcal{D}$  we have a natural injection

$$\mathcal{O}_{\mathcal{D},x} \longrightarrow K(\mathcal{D}).$$

We can then consider the following set:

$$\mathcal{D}_f = \{x \in \mathcal{D} : f_x \in \mathcal{O}_{\mathcal{D},x}\},$$

where  $\mathcal{O}_{\mathcal{D},x}$  is identified with its image in  $K(\mathcal{D})$ .

**Lemma 13.**  $\mathcal{D}_f$  is open.

*Proof.* Locally for every point  $x \in \mathcal{D}_f$ , we can write

$$f|_U = g/h$$

for some open affine  $U$ . Here  $h$  is not contained in the prime corresponding to  $x$ , by assumption on  $f$ . Let us consider the open subset  $D(h)$  in  $U$ . Then for every  $y \in D(h)$ , we see that  $f$  is contained in  $\mathcal{O}_{\mathcal{D},y}$ , as desired.  $\square$

By assumption on  $f$ , we see that any intersection point in the special fiber of  $\mathcal{D}$  lies in  $\mathcal{D}_f$  (We see that  $f$  is in fact invertible in  $\mathcal{O}_{\mathcal{D},x}$  for an intersection point  $x$ ). We then also see that any generic point  $y$  lying under  $x$  must also be an element of  $\mathcal{D}_f$ .

Since  $\mathcal{D}_f$  is open, for every  $x \in \mathcal{D}_f$  we can find an open affine  $U = \text{Spec}(A)$  such that for every  $\mathfrak{p} \in \text{Spec}(A)$  we have that  $f \in A_{\mathfrak{p}}$ . This then also means that  $f \in A$ . The ring

$$B = A[z]/(z^q - f)$$

is thus integral over  $A$  and is thus contained in  $A'$ .

**Lemma 14.** The algebra  $C = A[z][1/z]/(z^q - f)$  is standard étale over  $A$ .

*Proof.* We only have to check that the derivative of  $z^q - f$  with respect to  $z$ ,  $qz^{q-1}$ , is invertible in  $C$ . Since  $q$  is invertible by assumption on our rings and  $z$  is invertible by the localization we applied, we see that  $B$  is standard étale.  $\square$

Let us recall that for a morphism of schemes  $f : X \longrightarrow Y$  of finite type with  $Y$  locally Noetherian, we have the following equivalent statements:

1.  $f$  is étale at  $x \in X$ .
2.  $\hat{\mathcal{O}}_{X,x}$  is a free  $\hat{\mathcal{O}}_{Y,y}$ -module and  $\hat{\mathcal{O}}_{X,x}/\mathfrak{m}_y \hat{\mathcal{O}}_{X,x}$  is a finite separable field extension of  $k(y)$ . Here  $y = f(x)$ .

We furthermore have that if the induced map  $k(y) \longrightarrow k(x)$  is an isomorphism, then we have an isomorphism  $\hat{\mathcal{O}}_{X,x} = \hat{\mathcal{O}}_{Y,y}$ . All of this is contained in [[6], Prop 17.6.3]. Applying this to our situation, we have the following

**Lemma 15.** *Let  $\mathfrak{m}$  be a maximal ideal of  $A$  in  $D(f)$ . Then the induced morphism*

$$\hat{A}_{\mathfrak{m}} \longrightarrow \hat{B}_{\mathfrak{m}'}$$

*is an isomorphism. Here  $\mathfrak{m}'$  is a maximal ideal of the algebra  $B = A[z]/(z^q - f)$  lying above  $\mathfrak{m}$ .*

*Proof.* We note that since  $f \notin \mathfrak{m}'$  for any  $\mathfrak{m}'$  lying above it, we have that  $z \notin \mathfrak{m}'$  and that we thus have that the corresponding morphism of rings factors through the above standard étale algebra  $A[z][1/z]/(z^q - f)$ . We thus see that  $f$  is étale at  $\mathfrak{m}'$ . Furthermore, the residue fields of all these points are assumed to be algebraically closed, so we obtain an isomorphism of completions by the above considerations.  $\square$

We will mainly use this lemma for the intersection points in the intersection graph. Let us now explicitly compute some prime ideals. Let  $\mathfrak{p} \in \text{Spec}(A)$  correspond to a component in the special fibre and let  $\mathfrak{m} \in \text{Spec}(A)$  be a closed point in that component (i.e.  $\mathfrak{m} \supset \mathfrak{p}$ ).

**Lemma 16.** *The primes above  $\mathfrak{p}$  are given by*

$$\mathfrak{p}_i = \mathfrak{p} + \langle z - h_{\Gamma,i} \rangle.$$

*The maximal ideals above  $\mathfrak{m}$  are given by*

$$\mathfrak{m}' = \mathfrak{m} + \langle z - \alpha \rangle,$$

*where  $\alpha \in \mathcal{S}_x = \{\alpha \in k^* : \alpha^q = \overline{f}(x)\}$ .*

*Proof.* The primes above  $\mathfrak{p}$  correspond to primes of the ring

$$k(\mathfrak{p})[z]/(z^q - f),$$

where  $k(\mathfrak{p})$  is the residue field of  $\mathfrak{p}$ .

Writing out this correspondence for  $\mathfrak{p}$  yields the desired form as stated in the Lemma. One similarly proceeds for  $\mathfrak{m}$ .  $\square$

**Lemma 17.** *Let  $\mathfrak{m}$  be an intersection point in  $\mathcal{D}$  such that  $\hat{\mathcal{O}}_{\mathcal{D},x} \simeq R[[x,y]]/(xy - \pi^n)$ . The completion of  $(A[z]/(z^q - f))_{\mathfrak{m}'}$  with respect to  $\mathfrak{m}'$  is then also isomorphic to  $R[[x,y]]/(xy - \pi^n)$ .*

*Proof.* This follows from Lemma 15.  $\square$

**Corollary 6.1.**  $(A[z]/(z^q - f))_{\mathfrak{m}'}$  is normal.

*Proof.* First of all,  $A_{\mathfrak{m}}$  is an *excellent* ring. Furthermore, any finitely generated algebra over an excellent ring is again excellent and any localization of an excellent algebra is also excellent. We thus see that  $(A[z]/(z^q - f))_{\mathfrak{m}'}$  is excellent. We then use the following:

**Lemma 18** (*Normality of excellent rings*). *Let  $A$  be an excellent Noetherian local ring. Let  $\hat{A}$  be its formal completion. Then  $A$  is normal if and only if  $\hat{A}$  is normal.*

*Proof.* See [[9], page 344, Proposition 2.41].  $\square$

We thus see that  $(A[z]/(z^q - f))_{\mathfrak{m}'}$  is normal if and only if its completion is normal. Its completion is isomorphic to  $R[[x,y]]/(xy - \pi^n)$  for some  $n$  by Lemma 17, which is a normal ring. This gives the Corollary.  $\square$

We have thus identified the local rings of intersection points lying above an edge with maximal ideal  $\mathfrak{m}$ . We now link these maximal ideals to the components.

**Lemma 19.** *Let  $\mathfrak{m}'$  be an intersection point lying above  $\mathfrak{m}$ .*

1.  $\mathfrak{m}'$  uniquely corresponds to a solution  $\alpha \in k^*$  of the equation  $\alpha^q = \overline{f}(x)$ .
2. There exists a unique component  $\Gamma_i$  lying above  $\Gamma$  such that  $\mathfrak{m}'$  belongs to  $\Gamma_i$ .
3. For this component  $\Gamma_i$ , we have

$$\overline{h_{\Gamma,i}}(x) = \alpha.$$

*Proof.* Part (1) is just Lemma 16. We have that

$$\overline{h_{\Gamma}}(x)^q = \overline{f}(x) = \alpha^q,$$

so there exists a unique  $i$  such that

$$\overline{h_{\Gamma,i}}(x) = \alpha.$$

We then easily see that  $\mathfrak{m}'$  belongs to  $\Gamma_i$  and we automatically obtain part (3) of the Lemma.  $\square$

For every intersection point  $\mathfrak{m}'$  lying above  $\mathfrak{m}$ , we can now find a unique  $i$  and a unique  $j$  such that  $\mathfrak{m}'$  lies in both  $\Gamma_i$  and  $\Gamma'_j$ . We will therefore denote the maximal ideal  $\mathfrak{m}'$  lying above  $\mathfrak{m}$  by

$$\mathfrak{m}_{i,j} := \mathfrak{m}'.$$

**Corollary 6.2.**  $\Gamma_i$  and  $\Gamma'_j$  intersect each other in  $\mathfrak{m}_{i,j}$  with  $\Gamma_i \cdot \Gamma_j = 1$ .

*Proof.* This follows from Lemmas 17 and 19.  $\square$

Let us now fix a single  $i$  and  $j$  with components  $\Gamma_i$  and  $\Gamma'_j$ . We would now like to give a criterion for their intersection points.

**Lemma 20.**  $\Gamma_i$  and  $\Gamma'_j$  intersect each other if and only if there exists an intersection point  $x$  of  $\Gamma$  and  $\Gamma'$  such that

$$\overline{h_{\Gamma,i}}(x) = \overline{h_{\Gamma',j}}(x).$$

*Proof.* By Theorem 3.1, we see that the only intersections between  $\Gamma_i$  and  $\Gamma_j$  are those that are in the pre-image of an edge of  $\mathcal{G}(\mathcal{D})$ . We thus see that all intersections must arise from maximal ideals of the algebras

$$A_{\mathfrak{m}}[x]/(z^q - f),$$

where  $\mathfrak{m}$  corresponds to an intersection point  $x$  of  $\mathcal{D}_s$ . By Lemma 19, we now see that these maximal ideals are exactly given by equations of the form

$$\overline{h_{\Gamma,i}}(x) = \alpha = \overline{h_{\Gamma',j}}(x).$$

This then yields the Lemma.  $\square$

We thus see that we have a complete description of the intersection points lying above the intersection points of  $\Gamma$  and  $\Gamma'$ . In a concrete example, one has to do the following:

**[Procedure for étale morphisms of graphs]**

1. Determine local functions  $h_\Gamma$  and  $h_{\Gamma'}$  such that

$$\begin{aligned} (\overline{h_\Gamma})^q &= \overline{f}, \\ (\overline{h_{\Gamma'}})^q &= \overline{f}. \end{aligned}$$

For curves with genus  $g(\Gamma) > 0$ , this is quite easy to solve: there is only the possibility of constant functions. For curves with genus 0, one has to write the reduced form of  $f$  as a  $q$ -th power of some function.

2. Determine the values

$$\overline{h_\Gamma}(x)$$

for all intersection points  $x$  of  $\Gamma$  and  $\Gamma'$ . One then has to pair these values as in Lemma 20.

**Example 6.3.** Suppose we take the elliptic curve  $E$  defined by the equation

$$y^2 = x(x - \pi)(x + 1).$$

This has a 2-torsion point  $P = (-1, 0)$ . Indeed, one can easily check that

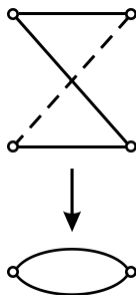


Figure 13: *The trampoline covering in Example 6.3.*

$$\text{div}_\eta(x + 1) = 2(P) - 2(\infty).$$

The elliptic curve has the semistable model obtained by

$$xt = \pi$$

and normalizing. The resulting equation is

$$\left(\frac{y}{x}\right)^2 = (1 - t)(x + 1).$$



which is easily seen to have two vertices and two edges. We take  $\Gamma = Z(x)$  and  $\Gamma' = Z(t)$ . We now take the normalization of this local model in the extension defined by

$$z^2 = x + 1.$$

One easily finds that the divisor  $x + 1$  is locally a square in  $E$ . Indeed, for  $t = 0$  we find that

$$\bar{y}^2 = \bar{x} + 1$$

and for  $x = 0$  we find that  $\bar{x} + 1 = 1$ . In our adopted notation we now have

$$\begin{aligned} h_{\Gamma,0} &= 1, \\ h_{\Gamma,1} &= -1, \\ h_{\Gamma',0} &= \bar{y}, \\ h_{\Gamma',1} &= -\bar{y}. \end{aligned}$$

Let us consider the intersection point  $\tilde{x}$  defined by  $x = 0 = t$  and  $\bar{y} = 1$ . We have

$$\begin{aligned} h_{\Gamma,0}(\tilde{x}) &= 1, \\ h_{\Gamma,1}(\tilde{x}) &= -1, \\ h_{\Gamma',0}(\tilde{x}) &= 1, \\ h_{\Gamma',1}(\tilde{x}) &= -1 \end{aligned}$$

We thus see that we have two intersection points lying above  $\tilde{x}$ : for the value 1  $\Gamma_0$  and  $\Gamma'_0$  intersect, for the value  $-1$  the components  $\Gamma_1$  and  $\Gamma'_1$  intersect. A similar computation gives the intersections lying above the other intersection point. We see that we obtain a *trampoline*-figure with 4 vertices, as obtained earlier. The covering can also be found in Figure 13. Since the obtained curve is again an elliptic curve, the reduction can also be obtained directly by calculating the reduction type.

**Example 6.4.** Suppose we take the same elliptic curve  $E$  with multiplicative reduction defined by

$$y^2 = x(x - \pi)(x + 1)$$

and suppose that we take a three torsion point  $P$  that does not reduce to  $(0, 0)$  (the singular point). We first find a function  $f$  such that

$$(f) = 3(P) - 3(\infty).$$

If we label the components of  $E$  as  $\Gamma$  and  $\Gamma'$  as before, we see that  $f|_{\Gamma}$  is a constant and that  $f|_{\Gamma'}$  is the cube of a nonconstant function  $\bar{h}_{\Gamma'}$ , which has a zero at  $\tilde{P}$  and a pole at  $\infty$ .

We have the numbers

$$\alpha_{h_{\Gamma',j}}(x) = \bar{h}_{\Gamma',j}(x)$$

for any intersection point. Note that for any  $j$  and  $x$  and  $y$  distinct intersection points, we have that  $\alpha_{h_{\Gamma',j}}(x) \neq \alpha_{h_{\Gamma',j}}(y)$  by the fact that  $P$  is a nontrivial torsion point in the identity component. We see that if we take the extension

$$z^3 = f,$$

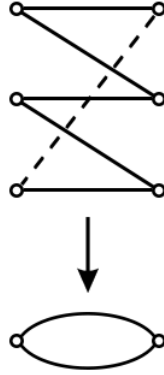


Figure 14: *The covering in Example 6.4.*

we obtain a reduction graph with 3 vertices above  $\Gamma$  and 3 above  $\Gamma'$ . By earlier considerations, we see that if we take any component  $\Gamma_0$  lying above  $\Gamma$ , it intersects *two distinct* other components lying above  $\Gamma'$ . The Galois group  $\mathbb{Z}/3\mathbb{Z}$  then cycles these intersections naturally to give 6 edges. By the calculation

$$e(E') - v(E') + 1 = 6 - 6 + 1 = 1,$$

we find that this graph has Betti number one, as expected. The covering of graphs can be found in Figure 14.

**Example 6.5.** Let us take the genus 2 curve  $C$  defined by

$$y^2 = x(x - \pi)(x + 1)(x + 1 - \pi)(x + 2)(x + 2 - \pi),$$

which has the usual reduction graph consisting of two vertices with three edges between them. We will however not take this model. As before, let

$$xt = \pi.$$

Then the normalization of this model in  $C$  is given by

$$(y/x)^2 = (1 - t)(x + 1)(x + 1 - \pi)(x + 2)(x + 2 - \pi).$$

For  $x = 0$  (with corresponding component  $\Gamma$ ), we obtain a single component, which we will call  $\Gamma_0$ . For  $t = 0$  (corresponding to  $\Gamma'$ ), we obtain two components  $\Gamma'_0$  and  $\Gamma'_1$  intersecting each other in two points. We also have that  $\Gamma_0$  intersects both  $\Gamma'_0$  and  $\Gamma'_1$  exactly once. We see that this yields a *subdivision* of the original intersection graph given by just taking the special fiber. This can also be seen in Figure 15. We now take the following divisor:  $D = (\pi, 0) - (0, 0)$ . This divisor cannot be principal because otherwise the curve would have genus 0. We have that

$$2D = \text{div}_\eta(f),$$

. where

$$f = \frac{x - \pi}{x} = 1 - t.$$

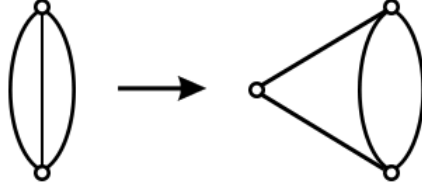


Figure 15: *The subdivision of the original graph in Example 6.5.*

It thus gives an element of  $J(C)$  that is 2-torsion.

For  $t = 0$ , we have that  $f$  is constant, whereas for  $x = 0$ , we have that  $f$  is the square of a nonconstant function. Namely, we have that

$$f|_{\Gamma_0} = \overline{(y/x)}^2.$$

We are thus in a similar situation as in the previous 2-torsion example. If we consider the extension

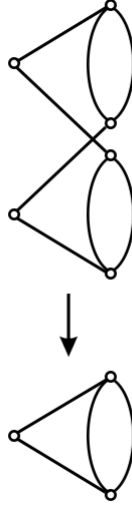


Figure 16: *The covering in Example 6.5.*

$$z^2 = f,$$

we see that we obtain 2 vertices lying above  $\Gamma_0$ :  $\Gamma_{0,0}$  and  $\Gamma_{0,1}$ . Similarly, we have 2 vertices lying above  $\Gamma'_0$  ( $\Gamma'_{0,0}$  and  $\Gamma'_{0,1}$ ) and  $\Gamma'_1$  ( $\Gamma'_{1,0}$  and  $\Gamma'_{1,1}$ ).

The intersection graph can now be found just as in the previous 2-torsion example. The only possible option up to relabeling components is:  $\Gamma_{0,0}$  intersects  $\Gamma'_{0,0}$  and  $\Gamma'_{1,1}$ ,  $\Gamma_{0,1}$  intersects  $\Gamma'_{0,1}$  and  $\Gamma'_{1,0}$ . Furthermore,  $\Gamma'_{0,0}$  intersects  $\Gamma'_{0,1}$  twice and  $\Gamma'_{1,0}$  intersects  $\Gamma'_{1,1}$  twice. This covering of

graphs can be found in Figure 16. This gives a graph with Betti number

$$e(\mathcal{C}) - v(\mathcal{C}) + 1 = 6 - 4 + 1 = 3,$$

as was to be expected from an unramified degree 2 covering of a genus 2 curve.

## 7 Graphs for Abelian Extensions of $\mathbb{P}^1$

In this section we will apply the theory from the earlier sections to the simplest case: abelian extensions of  $\mathbb{P}^1$ . We will first consider the easy case of hyperelliptic curves and then move over to more general abelian covers of  $\mathbb{P}^1$ . We will present a completely explicit exposition of the semistable models and the corresponding reduction graphs involved.

### 7.1 Suitable semistable models for $\mathbb{P}^1$

We will shortly review some material regarding certain semistable models for  $\mathbb{P}^1$  here, which we will view from a more or less analytic way. That is, we will give an interpretation using valuations (or absolute values) that is more suitable for tropical considerations. This material can also be found in [17] and [9] for instance.

So suppose we take the projective line with function field  $K(x)$ . Here  $K$  is again the quotient field of a discrete valuation ring  $R$  with maximal ideal  $\mathfrak{m}$  and residue field  $k$ . Let  $\mathcal{D}$  be a model for  $\mathbb{P}^1$  over  $R$ . Let  $D^0$  be the set of closed points of the generic fiber, which is just the set of closed points of  $\mathbb{P}^1$ . We then have a natural reduction map

$$r_{\mathcal{D}}(z) = \overline{\{z\}} \cap \mathcal{D}_s.$$

This map depends on the model  $\mathcal{D}$ . We can look at the formal fibers of this reduction map, given as

$$D_+(\tilde{z}) = \{z \in D^0 : r_{\mathcal{D}}(z) = \tilde{z}\}.$$

If  $\tilde{z}$  is a smooth point of  $\mathcal{D}_s$ , then we have

$$D_+(\tilde{z})(K) \simeq \mathfrak{m}_K.$$

If  $\tilde{z}$  is a double point we have for some  $n$  that the local completed ring (in  $\mathcal{D}$ ) is isomorphic to

$$R[[x, y]]/(xy - \pi^n).$$

We then have

$$D_+(\tilde{z})(K) \simeq \{a \in K : 0 < v(a) < n\}.$$

See [[9], page 469] for the proof.

We will now look at a very specific example that we will use very often. It is given by

$$D = \text{Proj}(R[X, Y, W]/(XY - \pi^n W^2)).$$

with the usual grading. It has an affine chart given by

$$U = \text{Spec}(R[x, y]/(xy - \pi^n)).$$

It has components  $\Gamma_0 = (x)$  and  $\Gamma_n = (y)$ . The labeling will be clear in a moment. We will identify  $K^*$  with its natural image in the generic fiber of  $\text{Spec}(R[x, t]/(xy - \pi^n))$ . This sends  $z$  to the prime ideal  $(x - z, zy - \pi^n)$ .

**Lemma 21.** *Let*

$$\begin{aligned} S_0 &= \{z \in K^* : v(z) \geq n\}, \\ S_n &= \{z \in K^* : v(z) \leq 0\}. \end{aligned}$$

*Then points in  $S_0$  and  $S_n$  reduce to smooth points in  $\Gamma_0$  and  $\Gamma_n$  respectively. Furthermore, 0 reduces to  $S_0$  and  $\infty$  reduces to  $S_n$ .*

*Proof.* Suppose that  $v(z) = n$ . Then  $z = \pi^n \cdot u$  and  $y = u$ , so  $z$  reduces to a nonsingular point. If  $v(z) > n$ , then  $z$  reduces to infinity. Indeed, in the patch

$$x' = \pi^n w^2,$$

where  $x' = \frac{X}{Y}$  and  $w = \frac{W}{Y}$  we find  $v(z') = v(z) - (n - v(z)) = 2v(z) - n$  and  $v(w) = v(z)$ . Thus  $z$  reduces to  $x' = 0$  and  $w = 0$ , corresponding to  $y = \infty$ . Similarly, if we take  $z = 0$ , then on the generic fiber we obtain the point  $x' = 0 = w$ .

A similar reasoning for  $v(z) \leq 0$  then gives that points of  $S_n$  reduce to nonsingular points of the component with  $y = 0$ . The point  $\infty$  then reduces to the point at infinity on that component, just as in the previous case. □

We would now like to have a subdivision for the rest of the points of  $K$  as well. To do this, we will blow-up the point  $\tilde{z} = (x, y)$ . We review the process. It is also described in [[9], page 366]. The blow-up of the point  $\tilde{z}$  induces a map  $U' \rightarrow U$ . We write

$$\begin{aligned} x_1 &= x/\pi, \quad y_1 = y/\pi, \\ y_2 &= y/x, \quad \pi_2 = \pi/x, \\ \pi_3 &= \pi/y, \end{aligned}$$

which are elements of the function field  $K(U)$ . The blow-up  $U'$  is then a union of three open affine subschemes  $U_i = \text{Spec}(A_i)$ . The first two algebras are

$$\begin{aligned} A_1 &= R[x, y, x_1, y_1] = R[x_1, y_1], \quad \text{with } x_1 y_1 = \pi^{n-2}, \\ A_2 &= R[x, y, y_2, \pi_2] = R[x, \pi_2], \quad \text{with } x \pi_2 = \pi, \end{aligned}$$

where  $y_2 \in R[x, \pi_2]$  because  $y_2 = y/x = \pi^n/x^2 = \pi_2^n x^{n-2}$  (note that these are switched in [9]). The last algebra is described similarly by

$$A_3 = R[y, \pi_3] \quad \text{with } y \pi_3 = \pi.$$

This blow-up now has three components. We will describe the transition functions that are relevant to us. We have an isomorphism

$$(A_1)_{x_1} \simeq (A_2)_{\pi_2},$$

sending  $x_1$  to  $\frac{1}{\pi_2}$  and  $y_1$  to  $\pi_2 \pi^{n-2}$ . Here the subscript means localization. This identifies the ideal  $(x) = (\frac{\pi}{\pi_2})$  with the radical of the ideal  $(y_1)$  in the special fiber. Similarly, we have an isomorphism

$$(A_1)_{y_1} \simeq (A_3)_{\pi_3}$$

that maps  $y_1$  to  $\frac{1}{\pi_3}$  and  $x_1$  to  $\pi^{n-2}\pi_3$ . This identifies the ideal  $(y) = (\frac{\pi}{\pi_3})$  with the radical of the ideal  $(x_1)$ .

In the algebra  $A_1$  we have another possible singular point. If so, we can perform more blow-ups until the algebras are all regular. The resulting *regular* scheme will be denoted by  $\tilde{U}$ . It has  $n + 1$  components. They will be labeled  $\Gamma_0, \Gamma_1, \dots, \Gamma_{n+1}$ .

**Lemma 22.** *Consider*

$$\begin{aligned} S_0 &= \{z \in K^* : v(z) \geq n\}, \\ S_1 &= \{z \in K^* : v(z) = n - 1\}, \\ &\vdots \\ S_{n-1} &= \{z \in K^* : v(z) = 1\}, \\ S_n &= \{z \in K^* : v(z) \leq 0\}. \end{aligned}$$

*Then the elements of  $S_i$  naturally reduce to smooth points of  $\Gamma_i$ .*

*Proof.* The lemma follows by repeatedly applying the reasoning of Lemma 21 to blow-ups of the algebra  $A_1 = R[x_1, y_1]/(x_1 y_1 - \pi^{n-2})$ .  $\square$

## 7.2 Hyperelliptic curves; examples

The author believes that it has been known for quite some time that the semistable reduction type of a curve  $C$  for an abelian cover

$$C \longrightarrow \mathbb{P}^1$$

greatly depends on the relative valuations of the branching points. This allows one in principle to create rather explicit "algorithms" that predict what kind of reduction type one obtains for these coverings. We will see in this section where these algorithms come from for the hyperelliptic case and explicitly calculate a lot of examples.

So let  $C \longrightarrow \mathbb{P}^1$  be a hyperelliptic covering. Using Kummer Theory, one sees that for  $\text{char}(K) \neq 2$  we have a function field extension given by the equation

$$y^2 = f(x),$$

where  $f(x)$  is a polynomial of a certain degree over the field  $K$ . Over a finite extension of  $K$ , we can now write  $f(x)$  as

$$f(x) = \prod_{i=1}^r (x - \alpha_i)$$

for certain elements  $\alpha_i \in \overline{K}$ . We assume that we have made the finite extension already and that  $\alpha_i \in K$ . For simplicity, we will now assume that  $v(\alpha_i) \geq 0$  for every  $i$ .

We will give a subproblem that will highlight most of the features of the general case. We will suppose that the branch locus is disjoint except perhaps for one point. This means that the reduction of the  $\alpha_i$  are disjoint except for a finite subset where we have  $\overline{\alpha_i} = \overline{\alpha_j}$ . We can safely assume that  $\alpha_i = 0$  for that finite subset. Let us calculate some semistable models for this case.

**Example 7.1.** We take the curve  $C$  defined by

$$y^2 = x(x - \pi)g(x), \quad (12)$$

where  $\pi$  is a uniformizer and  $g(x)$  is a polynomial of odd degree  $c = 2k + 1$  (the case of a polynomial with even degree is similar but with two points at infinity). We assume that the roots of  $g$  reduce to distinct points not equal to 0. We have

$$g(C) = k + 1.$$

Since the points  $(0)$  and  $(\pi)$  are not disjoint in the special fiber, we will want to create a semistable model for  $\mathbb{P}^1$  that makes them disjoint. We take:

$$\text{Proj} R[X, T, W]/(XT - \pi W^2)$$

with affine model

$$\text{Spec} R[x, t]/(xt - \pi).$$

We have that the point  $(0)$  on the generic fiber is now transferred to the affine part

$$\text{Spec} R[x', w]/(x' - \pi w^2),$$

where  $x' = \frac{X}{T}$  and  $w = \frac{W}{T}$ . Indeed, the corresponding prime ideal is  $(x', w)$ . The point  $(\pi)$  now corresponds to the prime ideal  $(x - \pi, t - 1)$  lying on the generic fiber. We see that the reductions of  $(0)$  and  $(\pi)$  now lie on the same component  $(x)$ , but they have distinct  $t$ -coordinates: one has  $t = 1$  and the other has " $t = \infty$ ".

We can thus use Theorem 1 and calculate the normalization of this scheme in the finite extension defined by equation (12). We'll first take a different route however, using only our knowledge of the divisors involved. Consider the divisor

$$\text{div}_\eta(f) = (0) + (\pi) + Z(g) - (2 + c) \cdot \infty.$$

We calculate

$$\rho(\text{div}_\eta(f)) = 2 \cdot (\Gamma_1) - 2 \cdot (\Gamma_2).$$

This means that the corresponding Laplacian function has slope  $\pm 2$  between  $\Gamma_1$  and  $\Gamma_2$ . The Laplacian can also be found in Figure 17. For the edge corresponding to the intersection  $\mathfrak{m} =$

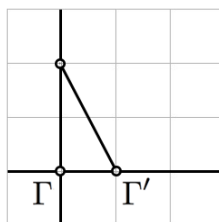


Figure 17: *The Laplacian of  $f$  in Example 7.1.*

$(x, t, \pi)$ , we thus obtain two edges in the pre-image.  
We now calculate  $f^{\Gamma_1}$ . We obtain

$$(f^{\Gamma_1}) = (x', w, \pi) + (x, t - 1, \pi) - 2 \cdot (\Gamma_1 \cap \Gamma_2).$$

If we thus consider the local equation

$$y^2 = f^{\Gamma_1},$$

it will ramify at 2 points. Thus the genus of the corresponding component above is 0. For  $\Gamma_2$  we have

$$(f^{\Gamma_2}) = Z(g) + 2 \cdot (\Gamma_1 \cap \Gamma_2) - (c + 2)(\infty).$$

Thus the equation  $y^2 = f^{\Gamma_2}$  ramifies in the points defined by  $Z(g)$  and  $\infty$ . There are  $c + 1$  of these, thus we can use the Riemann-Hurwitz formula to obtain

$$2g_{\Gamma_2} - 2 = 2(-2) + c + 1$$

and thus

$$g_{\Gamma_2} = k.$$

Thus the reduction graph consists of two vertices with two edges meeting them. The first component

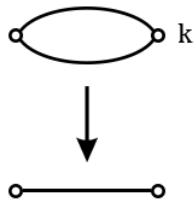


Figure 18: *The covering in Example 7.1.*

has genus 0 and the second component has genus  $k$ . The covering of graphs can be found in Figure 18.

We could have also calculated the normalization directly:

$$z^2 = (1 - t)g(x),$$

where  $z = y/x$ . Plugging in  $t = 0$  and  $x = 0$  then yields the same reduction graph.

**Example 7.2.** Let us take a slightly more involved example. We take

$$z^2 = x(x - \pi)(x - \pi^2)g(x),$$

where  $g(x)$  is a polynomial of even degree  $c = 2k$ . Then  $g(C) = k + 1$ . If we now consider the open affine defined by

$$R[x, y]/(xy - \pi^2),$$

then  $\pi$  does not reduce to a regular point. When we blow this point up, we obtain a new component where  $\pi$  does reduce to a regular point. The blow-up is given by the local charts





Figure 19: The Laplacian  $\phi$  of  $f$  in Example 7.2.

$$U_1 = \text{Spec}(R[x, t_2]),$$

$$U_2 = \text{Spec}(R[y, t_3]),$$

with relations

$$xt_2 = \pi, \tag{13}$$

$$yt_3 = \pi, \tag{14}$$

and the "obvious" local isomorphisms. We label the components  $Z(t_2) = \Gamma_1$ ,  $Z(x, y) = \Gamma_2$  and  $Z(t_3) = \Gamma_3$ . Here  $\Gamma_i$  intersects  $\Gamma_{i+1}$ .

We have

$$\text{div}_\eta(f(x)) = (0) + (\pi) + (\pi^2) + Z(g) - (c+3)(\infty),$$

where  $(0)$  and  $(\pi^2)$  reduce to  $\Gamma_3$  (the component with " $v(x) \geq 2$ "),  $(\pi)$  reduces to  $\Gamma_2$  and  $Z(g)$  and  $\infty$  reduce to  $\Gamma_1$ . Furthermore

$$\rho(\text{div}_\eta(f)) = 2(\Gamma_3) + (\Gamma_2) - 3(\Gamma_1),$$

whose Laplacian is depicted by a slope of 2 between  $\Gamma_3$  and  $\Gamma_2$  and a slope of 3 between  $\Gamma_2$  and  $\Gamma_1$ , as in Figure 19. Correspondingly, the edge  $e_{2,3}$  has two pre-images in  $\mathcal{C}$  and the edge  $e_{1,2}$  has one pre-image in  $\mathcal{C}$ . One finds that  $f^{\Gamma_1}$  has  $c+1$  ramification points and thus  $\Gamma'_1$  has genus  $k$  (Check this with the Riemann-Hurwitz formula). Similarly,  $f^{\Gamma_2}$  and  $f^{\Gamma_3}$  have two ramification points and as such they have genus 0. Thus the reduction graph consists of three vertices  $v_1, v_2, v_3$  where  $v_1$  and  $v_2$  intersect once and  $v_2$  and  $v_3$  intersect twice. The covering of graphs can be found in Figure 20.

We also give the normalizations for completeness. For the first chart  $U_1$  they are given by

$$\begin{aligned} z_1^2 &= x(1-t_2)(1-t_2\pi)g(x), \\ z_2^2 &= t_2(1-t_2)(1-t_2\pi)g(x), \\ z_1 \cdot z_2 &= \pi^{1/2}(1-t_2)(1-t_2\pi)g(x). \end{aligned}$$

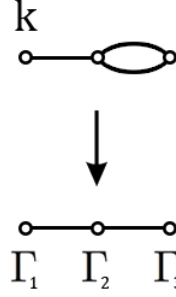


Figure 20: *The covering of graphs in Example 7.2.*

where  $z_1 = \frac{z}{x}$  and  $z_2 = \frac{t_2 z}{\pi^{1/2} x}$ . For the second chart  $U_2$  we have a single algebra given by

$$z_3^2 = (t_3 - 1)(1 - y)g(t_3 \cdot \pi),$$

where  $z_3 = \frac{z}{\pi^{1/2} t_3}$ . Note that in both charts we clearly see the need for the ramified extension of degree 2 given  $K \subseteq K(\pi^{1/2})$ .

### 7.3 Abelian coverings of $\mathbb{P}^1$

In the previous section, we saw that we can quite easily determine the reduction graph of a lot of hyperelliptic curves quite easily using divisors and their reductions. We will now state the process more generally for abelian covers of  $\mathbb{P}^1$ . The key step in this process will be the determination of the Laplacian of a certain defining function  $f$ . This Laplacian will determine most of the reduction graph, together with the rest of the branch points and their reductions.

So suppose we are given an abelian cover  $C \rightarrow \mathbb{P}^1$  of degree  $q$  over  $K$ . By Kummer Theory, we have that it is given by

$$y^q = f(x),$$

where  $f(x)$  possibly has multiple factors. We will give an algorithm that describes the reduction graph of  $C$ .

#### [Algorithm for semistable graphs of abelian coverings of $\mathbb{P}^1$ ]

1. Detect the branch points of the covering  $C \rightarrow \mathbb{P}^1$ . This is given as the zero set of some polynomial  $f$  with some points at infinity.
2. Find a semistable model  $\mathcal{D}$  of  $\mathbb{P}^1$  that separates the closure of the branch locus. Here the branch locus has to reduce to nonsingular points on the special fiber. This can be done by blowing up points in the special fiber.
3. Determine  $\rho(P)$  for any  $P \in \text{Supp}(f)$ . This is usually already done in the previous process.
4. Determine the Laplacian function  $\Delta(f)$ . This can be done by an iterative process which we will describe shortly.

5. Determine the  $g_e$  for an edge in the semistable model  $\mathcal{D}$ . This can be done using the Laplacian function.
6. Calculate the genera of the primes dividing the primes in the special fiber  $\mathcal{D}_s$ . This can be done using the Riemann-Hurwitz formula and the reduction of the ramification points.

**Example 7.3.** Let us do an example where our curve is given by a degree 3 covering of  $\mathbb{P}^1$ . Let's take the curve  $C$  given by the equation

$$z^3 = f(x) := x(x - \pi)g(x),$$

with  $c := \deg(g(x))$  and  $g(x)$  separable. We take  $g(x)$  such that  $\deg(g) + 2 \not\equiv 0 \pmod{3}$ . Then  $C$  has exactly one point at infinity and  $C$  ramifies above that point. We calculate

$$2g - 2 = -6 + 2 \cdot \text{Card}(\mathcal{R})$$

and

$$\text{Card}(\mathcal{R}) = c + 3,$$

so that

$$g(C) = c + 1.$$

We'll take the usual model with chart

$$R[x, t]/(xt - \pi).$$

As before, we have

$$\text{div}_\eta(f) = (0) + (\pi) + Z(g) - (2 + c) \cdot (\infty)$$

and

$$\rho(\text{div}_\eta(f)) = 2(\Gamma_1) - 2(\Gamma_2).$$

We thus see that the edge  $e = \Gamma_1 \cap \Gamma_2$  is preserved in  $\mathcal{C}_s$ , meaning that  $g_e = 1$ . We find that  $f^{\Gamma_1}$  has 3 ramification points and  $f^{\Gamma_2}$  has  $c + 2$ . Using the formula

$$g = -2 + \#\mathcal{R},$$

we find that  $\Gamma'_1$  has genus 1 and  $\Gamma'_2$  genus  $c$ . Thus the reduction graph consists of two vertices intersecting once, with weights 1 and  $c$ . The covering of graphs can be found in Figure 21.

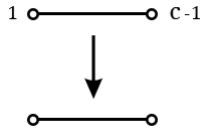


Figure 21: *The covering of graphs in Example 7.3.*

**Example 7.4.** Now take

$$z^3 = x(x - \pi)(x - 2\pi)g(x),$$

where  $c := \deg(g(x))$  is such that  $c + 3 \not\equiv 0 \pmod{3}$ . We then have that  $g(C) = c + 4 - 2 = c + 2$ . We take the model

$$R[x, t]/(xt - \pi).$$

with components  $Z(x) = \Gamma_1$  and  $Z(t) = \Gamma_2$ . Note that  $\Gamma_1$  corresponds to points with  $v(x) \geq 1$  and  $\Gamma_2$  to points with  $v(x) \leq 1$ . We thus find that  $(0), (\pi), (2\pi) \mapsto \Gamma_1$  and  $Z(g), (\infty) \mapsto \Gamma_2$ . We have

$$\operatorname{div}_\eta(f) = (0) + (\pi) + (2\pi) + Z(g) - (c + 3)(\infty)$$

with

$$\rho(\operatorname{div}_\eta(f)) = 3(\Gamma_1) - 3(\Gamma_2)$$

and thus  $g_e = 3$  for  $e = \Gamma_1 \cap \Gamma_2$ . The Laplacian can also be found in Figure 22. We see that

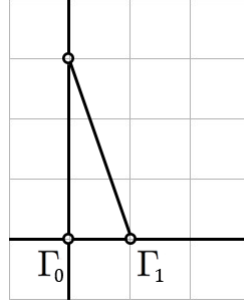


Figure 22: The Laplacian  $\phi$  of  $f$  in Example 7.4.

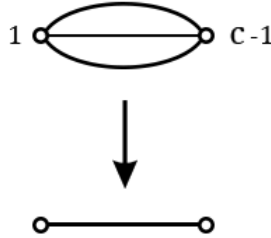


Figure 23: The covering of graphs in Example 7.4.

$$(f^{\Gamma_1}) = (\bar{0}) + (x, t - 1) + (x, t - 2) - 3(\Gamma_1 \cap \Gamma_2)$$

and thus that there are 3 ramification points. Thus  $g(\Gamma'_1) = 1$ . For  $\Gamma_2$  we have

$$(f^{\Gamma_2}) = Z(g) - (3 + c)(\infty) + 3(\Gamma_1 \cap \Gamma_2)$$

and thus  $g(\Gamma'_2) = c - 1$ . All in all, we can see that our reduction graph consists of 2 vertices with 3 edges between them. The corresponding weights on the vertices are  $c - 1$  and 1. The corresponding covering of graphs can be found in Figure 23.

Let us now say something about the Laplacian function. Suppose that we are given a semistable model  $\mathcal{D}$  of  $\mathbb{P}^1$  where the branch locus of  $C \rightarrow \mathbb{P}^1$  is separated in the special fiber. Suppose that we have already calculated the principal divisor

$$\Delta(f) = \sum_i c_i(\Gamma_i),$$

where the  $\Gamma_i$  are the components of  $\mathcal{D}_s$ . We start at components  $\Gamma_i$  with

$$\Gamma_i^2 = -1,$$

meaning that they only intersect one other component. The slope of the function is then given by  $c_i$ , so we draw a line of slope  $c_i$  towards the other component. By doing this for all leaves, one can continue to components lying in the interior. This process terminates exactly because  $\mathcal{G}(\mathcal{D})$  is a tree.

Now, let us suppose that we are only interested in the reduction graph *without* the weights of the genera of the components. We'll do an example where we only address this problem.

**Example 7.5.** Suppose we take something like

$$z^3 = x(x - \pi)(x - 2\pi)(x - \pi^2)(x - 2\pi^2)(x - \pi^3)g(x),$$

where  $c := \deg(g)$  is such that  $6 + c \not\equiv 0 \pmod{3}$ . We are now only interested in the *unweighted* graph corresponding to this curve. We create a semistable model with  $R[x, t]/(xt - \pi^3)$  blown up two times. We have 4 components  $\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3$  where  $\Gamma_0$  corresponds to  $v(x) \leq 0$ ,  $\Gamma_1$  to  $v(x) = 1$ ,  $\Gamma_2$  to  $v(x) = 2$  and  $\Gamma_3$  to  $v(x) \geq 3$ . We see that

$$\begin{aligned} 0, (\pi^3) &\mapsto \Gamma_3, \\ (\pi^2), (2\pi^2) &\mapsto \Gamma_2, \\ (\pi), (2\pi) &\mapsto \Gamma_1, \\ Z(g), (\infty) &\mapsto \Gamma_0. \end{aligned}$$

Our Laplacian is then

$$\rho(\operatorname{div}_\eta(f)) = 2(\Gamma_3) + 2(\Gamma_2) + 2(\Gamma_1) - 6(\Gamma_0).$$

The corresponding function has slope 6 from  $\Gamma_0$  to  $\Gamma_1$ , slope  $-4$  from  $\Gamma_1$  to  $\Gamma_2$ , slope  $-2$  from  $\Gamma_2$  to  $\Gamma_3$ , as in Figure 24. We thus see that

$$\begin{aligned} g_{\Gamma_0 \cap \Gamma_1} &= 3, \\ g_{\Gamma_1 \cap \Gamma_2} &= 1, \\ g_{\Gamma_2 \cap \Gamma_3} &= 1, \end{aligned}$$

which determines the graph. It is a graph with 4 neighbouring vertices, two of which have 3 edges between them. The covering of graphs can be found in Figure 25.

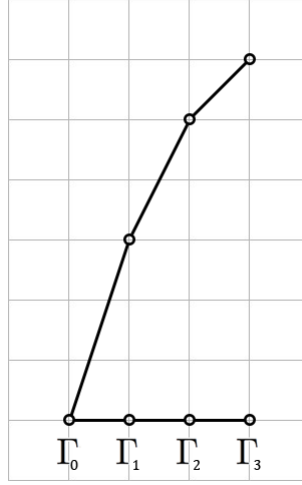


Figure 24: *The Laplacian  $\phi$  of  $f$  in Example 7.5.*

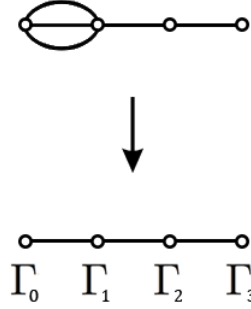


Figure 25: *The covering of graphs in Example 7.5.*

## 8 Graphs for solvable extensions

In this section we will study coverings  $C \longrightarrow D$  of curves that are Galois and have a *solvable* Galois group. We will also state a systematic way of obtaining the intersection graph for morphisms with a solvable Galois group. After that, we will turn to special cases, where mostly  $S_3$  will have our interest. This already gives a new proof of the semistability of elliptic curves. It also proves the criterion:  $v(j) < 0$  if and only if the elliptic curve has multiplicative reduction. We will then turn to genus 3 curves. We will exhibit a natural covering  $C \longrightarrow \mathbb{P}^1$  of degree 3, arising from the canonical embedding of  $C$  into  $\mathbb{P}^3$ . After that we will consider curves of genus 4, 5 and 6. These also have a natural morphism to  $\mathbb{P}^1$  with a solvable Galois group.

## 8.1 Solvable groups and coverings

Let  $G$  be a finite group. Then  $G$  is said to be *solvable* if there exists a finite chain of subgroups of  $G$ :

$$G_0 = (1) \subseteq G_1 \subseteq \dots \subseteq G_n = G,$$

where  $G_i$  is normal in  $G_{i+1}$  and  $G_{i+1}/G_i$  is abelian. An equivalent definition is that  $G$  admits a *composition series* such that every factor is cyclic of prime order. Recall that a composition series is a series

$$(1) = H_0 \triangleleft H_1 \triangleleft \dots \triangleleft H_r = G$$

such that each  $H_i$  is normal in  $H_{i+1}$  and  $H_{i+1}/H_i$  is simple.

We will now return to our usual setting of coverings. Suppose that  $C \rightarrow D$  is Galois with solvable Galois group. Then there exists a composition series

$$(1) = H_0 \triangleleft H_1 \triangleleft \dots \triangleleft H_r = G$$

with corresponding inclusions of function fields

$$K(C) \leftarrow (K(C))^{H_1} \leftarrow \dots \leftarrow (K(C))^{H_r} = K(D).$$

We define  $K(C_i) := (K(C))^{H_i}$ . Then for every inclusion  $H_i \triangleleft H_{i+1}$  we have a Galois extension

$$K(C_{i+1}) \rightarrow K(C_i)$$

that is cyclic of prime degree  $q$ . If our field  $K$  contains  $\zeta_q$  and is prime to the characteristic of our residue field, then this field extension can be described by an extension of the form

$$K(C_{i+1}) \rightarrow K(C_{i+1})[z]/(z^q - f_i) = K(C_i)$$

by Kummer theory.

Our strategy for solvable groups is now as follows. Suppose we have a morphism  $\mathcal{C} \rightarrow D$  that has a *solvable* Galois group.

### [Strategy for solvable groups]

- To obtain the intersection graph of  $\overline{\mathcal{C}}$ , one can do the following:

  1. Determine the Galois group  $G$  of the field extension  $K(\overline{\mathcal{C}})/K(D)$ .
  2. Find a composition series of  $G$ .
  3. Find the subextensions  $K(C_i)$  of  $K(\overline{\mathcal{C}})$  corresponding to the composition series.
  4. Use abelian methods on the morphisms  $C_i \rightarrow C_{i+1}$  to calculate the reduction graph of  $C_i$ , starting with  $C_r = D$ . This yields the intersection graph of  $\overline{\mathcal{C}}$ .

## 8.2 Algorithms for determining semistable intersection graphs

In this section, we will state some systematic procedures to determine intersection graphs for curves. We will start with abelian covers and then consider the problem of taking quotients.

So let us start by giving an algorithm for calculating the semistable intersection graph of a curve  $C$  with abelian cover  $\phi : C \rightarrow D$ . We assume that we already have the intersection graph of  $D$ .

[Algorithm for determining the intersection graph for abelian coverings  $C \rightarrow D$ ]

1. Determine the degree of the extension  $K(C) \supseteq K(D)$ .
2. Find an  $f$  such that the above field extension is given by

$$z^q = f.$$

3. Determine the divisor of  $f$ .
4. For every  $P \in \text{Supp}(f)$ , find  $\rho(P) \in \mathcal{G}(\mathcal{D})$ .
5. Find a solution  $\phi$  of  $\Delta(\phi) = \rho(\text{div}_\eta(f))$ .
6. Determine the divisors of the functions  $f^\Gamma$  using Propositions 5.7 and 5.8.
7. Determine the  $g_{\mathfrak{q}/\mathfrak{p}}$  using Propositions 5.4 and 5.5 and the  $a(\mathfrak{q})$  (the genus of the component corresponding to  $\mathfrak{q}$ ) using Propositions 5.7 and 5.8.
8. Determine if  $f$  gives rise to an étale morphism of graphs using Theorem 4.3 and Proposition 6.2. If it does, then one has to determine local functions  $h_\Gamma$  such that

$$(\overline{h_\Gamma})^q = \overline{f}$$

and one has to calculate for all intersection points in  $\mathcal{G}(\mathcal{D})$  the values

$$\overline{h_\Gamma}(x).$$

These then have to be paired as in Lemma 16 in section 6.2.1.

## 8.3 Special case: $S_3$

We will review the basic structure of  $S_3$ . It has a composition series

$$(1) \triangleleft A_3 \triangleleft S_3,$$

with quotients  $\mathbb{Z}/3\mathbb{Z}$  and  $\mathbb{Z}/2\mathbb{Z}$ . The way that  $S_3$  arises as the Galois group of a covering of a curve is as follows. Suppose that we are given a degree 3 extension  $C$  of a curve  $D$ . It is defined by a cubic polynomial  $f = z^3 + pz + q$  where  $p, q \in K(D)$ . We can take the discriminant  $\Delta(f)$  of  $f$  over  $K(D)$ . We have

$$\Delta(f) = 4p^3 + 27q^2.$$

We then have that  $K(C)/K(D)$  is Galois if and only if the discriminant is a square in  $K(D)$ . If  $K(C)/K(D)$  is Galois, then it is Galois of degree 3 and then we have an abelian extension  $K(C)/K(D)$ .



Suppose now that  $\Delta(f)$  is not a square. We then take the Galois closure of the field extension  $K(C)/K(D)$ . This is a field extension  $K(\overline{C}) \supseteq K(C) \supseteq K(D)$  such that  $K(\overline{C})/K(D)$  is Galois with Galois group  $S_3$ . It has an intermediate field corresponding to the normal subgroup  $A_3$ . This can be explicitly described by

$$K(C) \subseteq K(C)[y](y^2 - \Delta).$$

In other words, we take a square root of the discriminant. We denote this last field by  $K(\sqrt{\Delta})$ . We now have a cubic extension

$$K(\sqrt{\Delta}) \longrightarrow K(\overline{C})$$

that is actually Galois of degree 3 by Galois theory. Thus it is an abelian extension and assuming that  $\zeta_3 \in K$ , we have that it can be described by taking a cube root of some element. In the spirit of the famous Cardano formula, we find

$$K(\overline{C}) = K(\sqrt{\Delta})[w]/(w^3 - y + \sqrt{27}\Delta), \quad (15)$$

where the induced automorphism of order 2 from the subfield  $K(\sqrt{\Delta})$  is given on  $w$  by

$$\tau(w) = \frac{cw}{w}$$

where  $c$  is an element such that  $c^3 = -4$ . The automorphism of order 3 is of course given by  $w \mapsto \zeta w$ .

### 8.3.1 New proof of semistability of elliptic curves using nonabelian coverings

Let us see how one can use the solvability of  $S_3$  to once again study semistability of elliptic curves. Let us take an elliptic curve  $E$  over a field of characteristic not equal to 2 or 3. Then one can find an equation of the form

$$x^3 + Ax + B + y^2 = 0$$

(the unusual notation will become clear in a moment) for some  $A$  and  $B$  in  $K$ . Just as in [13], we can assume that  $v(A), v(B) \geq 0$ . To prove semistability of the curve, one usually considers the  $2 - 1$  covering given by

$$\phi(x, y) = x$$

and then uses the branch points to explicitly create the semistable model. We will make life hard for us now and consider a different covering:

$$\phi(x, y) = y.$$

This gives a degree 3 morphism  $E \rightarrow \mathbb{P}^1$  with corresponding extension of function fields  $K(y) \subset K(E)$ . We will first calculate the discriminant of this algebra. It is given by

$$\Delta = 4A^3 + 27(B + y^2)^2.$$

We would like to determine whether this is a square or not. To that end, we calculate the discriminant of

$$\Delta'(y_1) = 4A^3 + 27(B + y_1)^2$$

and see that

$$\Delta(\Delta'(y_1)) = (2 \cdot 27 \cdot B)^2 - 4 \cdot (27) \cdot (27B^2 + 4A^3) = -(4 \cdot 27)^2 \cdot A^3.$$

We therefore see that the discriminant  $\Delta$  is a square if and only if either  $A = 0$  or  $y = 0$  is a zero of  $\Delta$ . In the latter case we see directly that we must have  $4A^3 + 27B^2 = 0$ , which contradicts the assumption that  $E$  is nonsingular. The case  $A = 0$  is a separate case, where one can easily see that  $E$  has potential good reduction.

So let us assume that  $A \neq 0$ . Then the discriminant is not a square and we obtain a bonafide extension of degree two given by

$$z^2 = 4A^3 + 27(B + y^2)^2.$$

This is again a curve of genus 1, which we denote by  $E'$ . We would like to know the reduction type of this curve. We will do this in terms of the discriminant  $\Delta(E) = 4A^3 + 27B^2$ . We can scale the equation of  $E$  such that either  $v(A) = 0$  or  $v(B) = 0$ . This will generally require a finite extension of  $K$ .

We now consider the following possible scenarios for  $A, B$  and  $\Delta(E)$ :

1.  $v(A) = v(B) = 0$  and  $v(\Delta(E)) > 0$ .
2.  $v(A) = v(B) \geq 0$  and  $v(\Delta(E)) = 0$ .
3.  $v(A) > 0$ ,  $v(B) = 0$  and  $v(\Delta(E)) = 0$ .

One can quite easily see that these are mutually exclusive.

**Proposition 8.1.** Let  $\overline{E}$  be the Galois closure of the morphism  $\phi$ . Consider a triple  $(E, E', \overline{E})$  with corresponding intersection graphs

$$((\mathcal{G}(E), \mathcal{G}(E'), \mathcal{G}(\overline{E})).$$

Then every type of triple  $(v(A), v(B), v(\Delta(E)))$  (as described above) corresponds to exactly one triple of intersection graphs. The correspondence is given by

1. Suppose that  $v(A) = v(B) = 0$  and  $v(\Delta(E)) > 0$ . Then  $E'$  has multiplicative reduction,  $\mathcal{G}(\overline{E})$  consists of 3 copies of  $\mathcal{G}(E')$  meeting in one vertex and  $E$  has multiplicative reduction.
2. Suppose that  $v(A) = 0$ ,  $v(B) \geq 0$  and  $v(\Delta(E)) = 0$ . Then all curves involved are nonsingular.
3. Suppose that  $v(A) > 0$ ,  $v(B) = 0$  and  $v(\Delta(E)) = 0$ . Then  $E'$  has multiplicative reduction,  $\mathcal{G}(\overline{E})$  consists of two elliptic curves meeting twice.  $E$  has good reduction.

*Proof.* We subdivide the proof into three parts, according to the cases given in the statement of the proposition.

1. We first determine the reduction type of  $E'$ . We write

$$z^2 = 4A^3 + 27B^2 + 2 \cdot 27By^2 + 27y^4 = \Delta(E) + 2 \cdot 27By^2 + 27y^4.$$

In the case  $v(A) = v(B) = 0$  and  $v(\Delta(E)) > 0$ , we find that two of the roots of the polynomial on the righthand side coincide. We will not explicitly give them. Instead, we write

$$\Delta(E) = \pi^{2n}u,$$

for some  $u \in R^*$  (note that this usually requires a ramified extension of  $K$ ). We then take the semistable model obtained from the following

$$yt = \pi^n.$$

This then gives the equation

$$(z/y)^2 = t^2u + 2 \cdot 27B + 27y^2,$$

which can be seen to have two vertices (corresponding to  $y = 0$  and  $t = 0$ ) and two edges (corresponding to  $\frac{z}{y} =: z' = \pm\sqrt{2 \cdot 27B}$ ). We thus find that  $E'$  has multiplicative reduction.

We label the components by  $\Gamma_1 = Z(y)$  and  $\Gamma_2 = Z(t)$ .

Now for  $\overline{E}$ . The extension is given by the function

$$f = z - \sqrt{27}q = z - \sqrt{27}(y^2 + B).$$

We first have to find the divisor of this function on the generic fiber. This is given by

$$\operatorname{div}_\eta(f) = 2 \cdot (\infty_1) - 2 \cdot (\infty_2),$$

where the  $\infty_i$  are the two points at infinity. In the case we are in, these points both reduce to nonsingular points on  $\Gamma_2$ . We thus find that  $\operatorname{div}(f^{\Gamma_1}) = (1)$  and

$$\operatorname{div}(f^{\Gamma_2}) = 2 \cdot \overline{\infty_1} - 2 \cdot \overline{\infty_2}.$$

Thus  $\Gamma_1$  has three pre-images and  $\Gamma_2$  has one pre-image, where  $\Gamma'_2 \rightarrow \Gamma_2$  ramifies at two points, giving  $g(\Gamma'_2) = 0$ . The reduction graph is thus of the indicated type.

The action of  $\tau$  on the edges and vertices is not directly obvious. We use some Galois theory to determine the action. Suppose we take a prime  $\mathfrak{q}$  dividing  $\mathfrak{p}$ , where  $\mathfrak{p}$  corresponds to  $\Gamma_1$ . There are three extensions of  $\mathfrak{q}$ , thus we have that

$$\operatorname{Card}(D_{\mathfrak{q}/\mathfrak{p}}) = 6/3 = 2.$$

There thus exists a nonnormal subgroup of order 2 that fixes  $\mathfrak{q}$ . For every other prime  $\mathfrak{q}'$

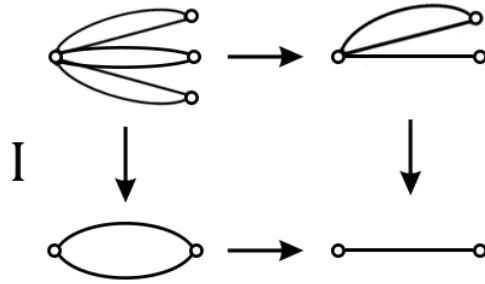


Figure 26: *The Galois closure of graphs in Case I.*

dividing  $\mathfrak{p}$ , we have that the decomposition group of  $\mathfrak{q}$  is conjugated to that of  $\mathfrak{q}'$ .

For the only prime  $\mathfrak{q}$  dividing the prime  $\mathfrak{p}$  corresponding to  $\Gamma_2$ , we have that the decomposition group has order 6. Indeed,  $S_3$  acts as an automorphism group on the corresponding extension of function fields.

We now turn to the edges. There exists only one edge in  $\mathcal{D}$ , so we call the prime corresponding to it  $\mathfrak{p}$ . There exist 6 extensions of  $\mathfrak{p}$  to  $\bar{\mathcal{E}}$ , so the decomposition group is (1). That is, there is no element of  $S_3$  fixing any of the edges.

If we now want to divide out the subgroup  $H = \langle \tau \rangle$  for  $H$  corresponding to  $E$ , we find that it fixes two vertices. In the quotient, these are linked by one edge. For the other two vertices, the action is transitive on both the edges and vertices connecting to the "center" vertex. Thus we obtain the following graph from left to right: a vertex intersecting another vertex twice, and then another vertex intersecting the prior vertex once. The entire Galois diagram with all the coverings can be found in Figure 26. We directly see that this intersection graph is semistable with  $\beta(\mathcal{G}(\mathcal{E})) = 1$ , thus  $E$  has multiplicative reduction.

2. A quick calculation shows that all curves in sight are nonsingular. We thus obtain trivial graphs with weights 1, 3, 1. Hence  $E$  has good reduction. The corresponding coverings of graphs can be found in Figure 27.

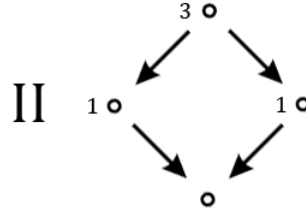


Figure 27: *The Galois closure of graphs in Case II.*

3. Suppose that  $v(A) > 0$ ,  $v(B) = 0$  and  $v(\Delta(E)) = 0$ . <sup>(3)</sup> There is no need for a special semistable model of  $\mathbb{P}^1$ : we directly have two components  $\Gamma_1$  and  $\Gamma_2$  that correspond to the equations:

$$\begin{aligned} z &= \sqrt{27}(B + y^2), \\ z &= -\sqrt{27}(B + y^2). \end{aligned}$$

These two components intersect in the two points defined by

$$y^2 = -B, z = 0.$$

Thus the intersection graph of  $E'$  consists of two vertices and two edges. We now have to know the reduction of  $\text{div}_\eta(y - \sqrt{27}(q)) = 2(\infty_1) - 2(\infty_2)$ . We quite easily check that

$$\infty_i \mapsto \Gamma_i.$$

---

<sup>3</sup>The author has to confess that this feels like we're using too much machinery, because we already know that  $E$  has good reduction from the fact that the reduced discriminant is nonzero. Nonetheless, calculating the entire Galois closure shows some interesting features.

Thus the Laplacian takes the form

$$\rho(\operatorname{div}_\eta(f)) = 2(\Gamma_1) - 2(\Gamma_2).$$

Writing  $f := y - \sqrt{27}q$ , we find that  $f^{\Gamma_i}$  has two "vertical" ramification points corresponding

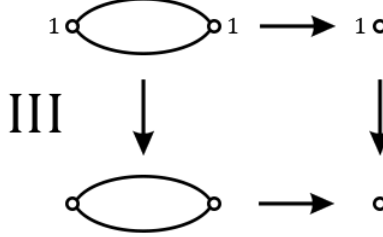


Figure 28: *The Galois closure of graphs in Case III.*

to the edges  $e_1$  and  $e_2$ . Furthermore, for every  $i$ , we have one point reducing to that component, so there is an extra ramification point. All in all, we find that for every component we have an extension that ramifies in three points, and as such we have that  $g(\Gamma'_i) = 1$ . The reduction graph is thus defined by two vertices with weights 1 and two edges between them. The action of  $\tau$  on the components sends  $\Gamma'_1$  to  $\Gamma'_2$ , so we find that the reduction graph of  $E$  consists of a single vertex with weight 1. The Galois diagram containing all the coverings can be found in Figure 28.

□

**Corollary 8.1.** An elliptic curve  $E$  has multiplicative reduction if and only if  $v(j) < 0$ .

*Proof.* Every elliptic curve can be put in one of the three scenarios considered in Proposition 8.1, with corresponding values of  $A$ ,  $B$  and  $\Delta$ . Case 1 corresponds exactly to  $v(j) < 0$  by the calculation  $v(j) = v(1728 \cdot \frac{4A^3}{\Delta}) = -v(\Delta) < 0$ . Cases 2 and 3 then naturally correspond to  $v(j) \geq 0$ , giving the Corollary. □

## 8.4 Genus 3 curves

We now turn to genus 3 curves. For genus 3 curves, we have that the moduli space of isomorphism classes has dimension 6 (in general,  $\mathcal{M}_g$  is irreducible of genus  $3g - 3$ ). If we look at the subspace of all hyperelliptic curves of genus 3, one quickly finds that this space has dimension 5. The idea is that for  $\operatorname{char} K \neq 2$ , one can locally write such a curve as

$$y^2 = x(x-1)(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)(x-\alpha_5)$$

by putting three of the ramification points of the hyperelliptic involution at  $\{0, 1, \infty\}$ . We thus find that not all curves of genus 3 have a hyperelliptic involution. So a different strategy is needed here. We will soon find out that one can in fact find a morphism of degree 3 to  $\mathbb{P}^1$  for curves that are not hyperelliptic. Such a morphism need not be Galois however, so we take the Galois closure of

this morphism. This will lead to a subcurve of genus  $\leq 4$  which is hyperelliptic plus an abelian cover of degree 3 of this curve which has genus  $\leq 12$ . The quotient of this graph by the involution of degree 2 gives the semistable graph of  $C$ .

We will see quite quickly that the Galois subextension of degree 3 is quite often *unramified*. This means that it comes from a 3-torsion point in the Jacobian. This now also means that we need to consider various abelian étale covers of graphs.

### 8.4.1 Preliminaries

Suppose we take a nonhyperelliptic curve of genus 3. By Riemann-Roch, we find that the canonical divisor on  $C$  defines a canonical embedding

$$C \longrightarrow \mathbb{P}^2,$$

which has degree 4, meaning that it is a nonsingular quartic. Conversely, every nonsingular quartic defines a nonhyperelliptic curve of genus 3. We now take a point  $P$  on  $C$  (which might need a finite extension of  $K$ ). Consider the space of all lines intersecting  $P$ . This is isomorphic to  $\mathbb{P}^1$ . If we now take any other point  $Q \in C(\overline{K})$ , we have that there is a line intersecting  $Q$  and  $P$ . We define

$$\phi(Q) = L_{P,Q},$$

where  $L_{P,Q}$  is the line connecting the two points. This defines a morphism  $\phi : C \longrightarrow \mathbb{P}^1$  of degree 3, since any hyperplane section intersects  $C$  in four points and we already have  $P$  as an intersection point.

**Example 8.1.** As an example, take the plane curve defined by

$$x^4 + y^4 - 1 = 0.$$

It has the rational point  $P = (1, 0)$ . Consider all lines of the form

$$y = t(x - 1).$$

By plugging this in, we obtain

$$x^4 - 1 + t^4(x - 1)^4 = 0$$

and thus

$$(x - 1)(x^3 + x^2 + x + 1 + t^4(x - 1)^3) = 0.$$

We cancel out  $x - 1$  (the obvious intersection point), to obtain

$$x^3 + x^2 + x + 1 + t^4(x - 1)^3 = 0.$$

This curve has an obvious degree 3 morphism to  $\mathbb{P}^1$ , given locally by

$$(x, t) \longmapsto t.$$

Let us now return to the general case. We have a morphism of degree 3

$$\phi : C \longrightarrow \mathbb{P}^1.$$

We now wish to arrive at some kind of "normal form". We take any quartic

$$f(x, y) = \sum_{i,j} c_{i,j} x^i y^j$$

and assume by translating that  $P = (0, 0)$  lies on  $C$ . Thus  $c_{0,0} = 0$ . Write

$$y = tx.$$

We then obtain

$$f(x, tx) = \left( \sum_{i,j} c_{i,j} x^i (tx)^j \right).$$

Canceling  $x$ , we thus obtain an equation of the form

$$f'(x, t) = \sum_{j=0}^{j=3} a_j(t) x^j = 0,$$

where  $\deg(a_j(t)) \leq j + 1$ . We find that

$$f'(x - \frac{a_2(t)}{3a_3(t)}, t) = a_3 x^3 + ((-3a_2^2 + 9a_1 a_3)/(9a_3))x + (2/3 \cdot a_2^3 - 3a_1 a_2 a_3 + 9a_0 a_3^2)/(9a_3^2).$$

Multiplying by  $a_3^2$  and taking  $x' = a_3 \cdot x$ , we obtain

$$f''(x', t) = x^3 + ((-3a_2^2 + 9a_1 a_3)/(9))x + (2/3 \cdot a_2^3 - 3a_1 a_2 a_3 + 9a_0 a_3^2)/(9).$$

We define

$$\begin{aligned} p(t) &= ((-3a_2^2 + 9a_1 a_3)/(9)), \\ q(t) &= (2/3 \cdot a_2^3 - 3a_1 a_2 a_3 + 9a_0 a_3^2)/(9), \end{aligned}$$

and see that

$$\Delta := (4p^3 + 27q^2)/(a_3)^2 = -a_1^2 a_2^2 + 4a_2^3 a_0 + 4a_1^3 a_3^1 - 18a_1 a_2 a_0 a_3^1 + 27a_0^2 a_3^2.$$

**Lemma 23.** *For each monomial  $m$  in  $\Delta$ , we have that  $\deg(m) \leq 10$ .*

*Proof.* This follows quite easily from  $\deg(a_j) \leq j + 1$  and some easy calculations on  $\Delta$ .  $\square$

We can now explicitly describe the curves in the Galois closure. For the quadratic subfield, we have that it is given by

$$y^2 = \Delta.$$

The cubic extension of this field is then given by

$$w^3 = y - \sqrt{27}q(t).$$

**Lemma 24.** *We have that*

$$\begin{aligned} g(\overline{C}) &\leq 12, \\ g(D) &\leq 4. \end{aligned}$$

*Proof.* From Lemma 23 we see that the degree of  $\Delta$  is at most 10, and as such we see that the genus of the corresponding curve  $y^2 = \Delta$  can be at most 4.

For  $\overline{C}$ , we use Riemann-Hurwitz on the covering  $\phi_3 : \overline{C} \rightarrow D$ . Let us first describe the ramification of the degree 3 morphism  $\phi_3$ . Recall that it is given by

$$w^3 = y - \sqrt{27}q(t).$$

Let  $f := y - \sqrt{27}q(t)$ . Then the ramification points are exactly the points where  $v_P(f)$  is not divisible by 3. Using the relation

$$(y - \sqrt{27}q(t))(y + \sqrt{27}q(t)) = 4p(t)^3,$$

one sees that the valuation is always divisible by 3 in this affine patch. The only points of ramification are thus the point(s) at infinity. How many there are of these, depends on the degree of the squarefree part of  $\Delta$ . Indeed, if the squarefree part of  $\Delta$  has *even* degree, then  $D$  has two points at infinity and if it has *odd* degree, then it has exactly one point at infinity.

Rewriting the Riemann-Hurwitz formula for the covering  $\phi_3$ , we obtain

$$g_{\overline{C}} - 1 = 3(g_D - 1) + \#(R).$$

The maximal occurring  $g_D$  and  $\#(R)$  are respectively 4 and 2, so we obtain that

$$g_{\overline{C}} \leq 9 + 2 + 1 = 12,$$

as desired. □

**Lemma 25.** *There are 8 options for  $(g(D), \#R, g(\overline{C}), \#(R_{\overline{C}/C}))$ . They are given by*

$g(D)$	$\#(R)$	$g(\overline{C})$	$\#(R_{\overline{C}/C})$
2	1	5	0
2	2	6	2
3	0	7	4
3	1	8	6
3	2	9	8
4	0	10	10
4	1	11	12
4	2	12	14

*Proof.* One uses the Riemann Hurwitz on both the covering  $\overline{C} \rightarrow D$  and  $\overline{C} \rightarrow C$ . This leads to

$$\begin{aligned} g_{\overline{C}} - 1 &= 3(g_D - 1) + \#(R), \\ 2g_{\overline{C}} - 2 &= 2(2g_C - 2) + \#(R_{\overline{C}/C}), \end{aligned}$$

where  $g_C = 3$ . Plugging in the possible values for  $D$  and  $\#(R)$  yields the above values. □



Let us now find the intersection graph of a genus 3 curve by a nonabelian morphism  $C \longrightarrow \mathbb{P}^1$  to illustrate the above.

**Example 8.2.** Let us consider the genus 3 curve defined by

$$z^3 + x^3 z + (x^3 + \pi^3) = 0.$$

We can find the reduction type in two ways: via an abelian cover and a nonabelian cover. The abelian cover is given by

$$(x, z) \longmapsto z$$

and the nonabelian one by

$$(x, z) \longmapsto x.$$

We will only consider the nonabelian cover. Note that we have  $p = x^3$  and  $q = x^3 + \pi^3$ , so that

$$\Delta = 4p^3 + 27q^2 = 4x^9 + 27 \cdot (x^3 + \pi^3)^2.$$

The corresponding quadratic extension is then given by

$$y^2 = \Delta,$$

which gives a hyperelliptic genus 4 curve  $D$ .

Taking the model with

$$xt = \pi,$$

we see that by normalizing we get a local model for  $D$  given by

$$(y/x^3)^2 = 4x^3 + 27(1 + t^3)^2,$$

which has three components in the special fiber. Above  $x = 0$ , we find two components  $\Gamma_{-1}$  and  $\Gamma_1$  given by the equations

$$y' = \pm\sqrt{27}(1 + t^3).$$

Above  $t = 0$  we find an elliptic curve (labeled by  $\Gamma'$ ) with corresponding equation

$$y'^2 = 4x^3 + 27.$$

We have three edges between  $\Gamma_{-1}$  and  $\Gamma_1$  (given by  $t^3 + 1 = 0$ ), one between  $\Gamma_{-1}$  and  $\Gamma'$  and another one between  $\Gamma_1$  and  $\Gamma'$ . This gives the reduction graph of  $D$ . It can be found in Figure 29. We now

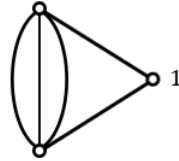


Figure 29: *The intersection graph of the intermediate genus 4 curve in Example 8.2.*

examine the divisor  $f = y - \sqrt{27}q$ . Since  $D$  has only one point at infinity, there is only 1 possible

ramification point. We quite quickly see that the valuation of  $f$  at infinity is divisible by 3, so the covering  $\bar{C} \rightarrow C$  is unramified everywhere. It therefore comes from a 3-torsion point.

The support of  $f$  is given by the points  $P_1 = (x, y - \sqrt{27}\pi^3)$  and  $P_2 = \infty$ . We see that

$$\operatorname{div}_\eta(f) = 9(P_1) - 9(P_2).$$

We therefore actually have a 9-torsion point. The divisor we are interested in is  $D = 3P_1 - 3P_2$  (which gives the extension). We find

$$\begin{aligned}\rho(P_1) &= \Gamma_1, \\ \rho(P_2) &= \Gamma'.$$

We first want to clarify one thing: when writing down the reduction graph, one needs to keep in mind the lengths of the corresponding edges. For every edge between  $\Gamma_{-1}$  and  $\Gamma_1$  for instance we have that the edge has length 3, which can be seen by the relation

$$(y' - \sqrt{27}(1 + t^3))(y' + \sqrt{27}(1 + t^3)) = 4\pi^3/(t^3)$$

(and the fact that  $t$  is invertible at these intersection points). The other two edges have length 1. We can now find a solution for the Laplacian. One of them is given by

$$\begin{aligned}\phi(\Gamma') &= 0, \\ \phi(\Gamma_-) &= 3, \\ \phi(\Gamma_+) &= 6.\end{aligned}$$

The corresponding graph of the Laplacian can be found in Figure 30. Note that the increase of

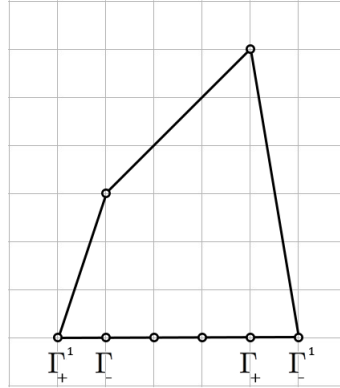


Figure 30: *The Laplacian function  $\phi$  corresponding to  $f$  in Example 8.2. There are three edges between  $\Gamma_-$  and  $\Gamma_+$ , but the Laplacian on each is the same.*

slope for every edge between  $\Gamma_-$  and  $\Gamma_+$  is taken to be 1, so that the total increase from  $\Gamma_-$  to  $\Gamma_+$  is 3.

Let us now consider the extension

$$w^3 = f.$$

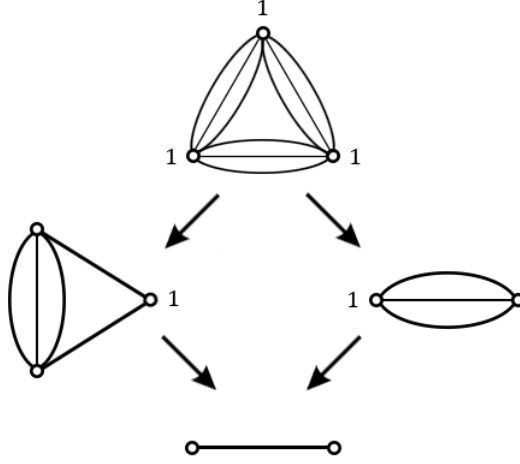


Figure 31: *The Galois closure of graphs in Example 8.2.*

The corresponding curve  $\overline{C}$  has genus 10. For  $f^{\Gamma_+}$  and  $f^{\Gamma_-}$ , we see that there are 3 ramification points, because the slope between them on the three edges is not divisible by 3. The slope of  $\phi$  between  $\Gamma_+$  and  $\Gamma'$  for instance is  $-6$ , so  $f^{\Gamma_+}$  does not ramify at that intersection point. We therefore have that there are two components  $\Gamma_{+,0}$  and  $\Gamma_{-,0}$  above  $\Gamma_+$  and  $\Gamma_-$  respectively. The corresponding morphism on the special fiber is ramified at exactly 3 points and so these components are genus 1 curves. On the other 2 edges, we have that the slope of  $\phi$  is divisible by 3, so there are 3 edges lying above them. On  $\Gamma'$ , it just defines an unramified extension of an elliptic curve, so we have one component which we call  $\Gamma'_0$  with genus 1 again.

We obtain the following reduction graph. We have three vertices. Each of these vertices intersects the other vertex in exactly three edges. Furthermore, these vertices all have weights 1. The intersection graph can be found in Figure 31.

Let us now consider the Galois action of  $\tau$  on this graph. Note that for the intersection graph of  $D$ , we have that  $\tau$  is trivial on all the edges. In the quotient, we have that these edges become smooth points. This happens because the morphism we created from  $D$  to  $\mathbb{P}^1$  is not disjointly branched. We have that  $\tau$  fixes  $\Gamma'_0$  and switches the other two vertices  $\Gamma_{+,0}$  and  $\Gamma_{-,0}$ . One can see this using the fundamental equality

$$n = e_p f_p g_p$$

from equation 1 in section 3.3. This then gives that the decomposition group of  $\Gamma'_0$  is  $S_3$  and the decomposition group of  $\Gamma_{+,0}$  and  $\Gamma_{-,0}$  are both the normal subgroup  $\mathbb{Z}/3\mathbb{Z}$ . The reduction graph in the quotient is then a graph on two vertices, intersecting each other in 3 edges. One of these vertices has weight 1 and is obtained as the quotient of  $\Gamma_{+,0}$  and  $\Gamma_{-,0}$ . The corresponding Galois diagram can be found in Figure 31.

**Example 8.3.** Consider the curve  $C$  defined by

$$z^3 + p(x)z + q(x) = 0$$

with

$$\begin{aligned} p(x) &= x^3, \\ q(x) &= x^4 + \pi^4. \end{aligned}$$

This is again a genus 3 curve with a nonabelian morphism

$$\phi : (x, z) \mapsto x$$

of degree 3. The intermediate curve  $D$  defined by

$$y^2 = 4p^3 + 27q^2 = 4x^9 + 27(x^4 + \pi^4)$$

then has genus 4 as before. Taking the semistable model corresponding to  $xt = \pi$ , we obtain the equation

$$(y')^2 = 4x + 27(1 + t^4)^2$$

with  $y' = y/x^4$ . This has the following reduction graph: we have two vertices above  $x = 0$ , corresponding to

$$(y') = \pm\sqrt{27}(1 + t^4).$$

These components  $\Gamma_+$  and  $\Gamma_-$  intersect each other 4 times, corresponding to the roots of  $1 + t^4$ . Above  $t = 0$  we have one component  $\Gamma_0$  of genus 0.

We now check the divisor of  $f = y - \sqrt{27}q$  as before. We again find

$$\text{div}_\eta(f) = 9(P_1) - 9(\infty),$$

where  $P_1 = (x, y - \sqrt{27}\pi^4)$ . Note that  $P_1$  reduces to  $\Gamma_+$ . We then again have the divisor on graphs

$$\rho(\text{div}_\eta(f)) = 9\Gamma_+ - 9\Gamma_0.$$

Note that the length of every edge in  $\mathcal{G}(\mathcal{D})$  is 1 by the identity

$$(y' - \sqrt{27}(1 + t^4))(y' + \sqrt{27}(1 + t^4)) = 4\pi/t$$

and the fact that  $\phi_{\mathcal{G}}$  is étale above the edge corresponding to  $x = t = 0$ . We then find the following Laplacian as a solution:

$$\begin{aligned} \phi(\Gamma_0) &= 0, \\ \phi(\Gamma_+) &= 4, \\ \phi(\Gamma_-) &= 5. \end{aligned}$$

See also Figure 32. Note that the slope between every pair of vertices is *not* divisible by 3. We therefore find that the reduction graph of the Galois closure has the same reduction graph as  $\mathcal{D}$ , but with different weights.

For  $\Gamma_-$  and  $\Gamma_+$ , we find that they both have 5 branch points (corresponding to the intersection points), so that their genera are 3. For  $\Gamma_0$ , we find that  $g(\Gamma_0) = 0$ . This determines the reduction graph.

If we now take the invariants under the automorphism of order  $\tau$  corresponding to  $C$ , we then obtain the graph consisting of two vertices, with one vertex having genus 3 and the other having genus 0. We thus see that  $C$  has potential good reduction. For the Galois diagram of graphs, see Figure 33.

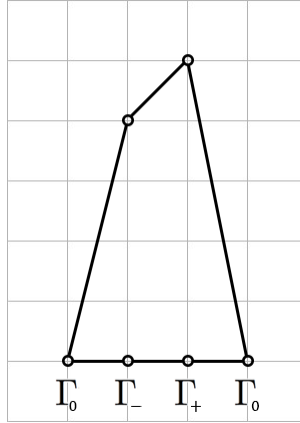


Figure 32: *The Laplacian in Example 8.3.*

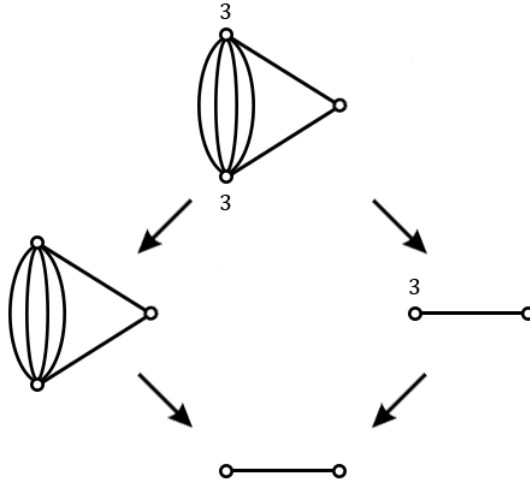


Figure 33: *The Galois closure of graphs in Example 8.3.*

## 8.5 Genus 4,5 and 6

Let us now quickly say something about curves of higher genus. We will adopt the notation from [18]. We say that a curve has a  $g_d^1$  if there exists a linear system of degree  $d$  and dimension 1 on  $C$ . This then automatically gives a morphism

$$\phi : C \longrightarrow \mathbb{P}^1$$

of degree  $d$ . We will state the result in [[18], page 345, Remark 5.5.1] again.

**Proposition 8.2.** For any  $d \geq \frac{1}{2}g + 1$ , any curve of genus  $g$  has a  $g_d^1$ . For  $d < \frac{1}{2}g + 1$ , there exist

curves of genus  $g$  having no  $g_d^1$ .

Let us now set  $d = 4$ . We will explain why in a moment. At any rate, we then find that any curve of genus 4, 5, 6 admits a  $g_4^1$ . Furthermore, for higher genus there exist curves having no  $g_4^1$ . For any curve of genus 4, 5 or 6, we thus have a morphism of degree 4

$$\phi : C \longrightarrow \mathbb{P}^1.$$

The Galois group of this morphism is then a subgroup of  $S_4$ . Since  $S_4$  is solvable, we can again use our techniques to find the reduction type of any curve of genus 4, 5 or 6.

Problems arise for our method for Galois morphisms

$$C \longrightarrow \mathbb{P}^1$$

that have  $A_5$  as its Galois group. The techniques developed in this paper are then no longer applicable, since  $A_5$  is not solvable. If one can find a different morphism that has a solvable Galois group, then one can still find the reduction type of  $C$ .

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